BARGAINING IN DYNAMIC MARKETS WITH MULTIPLE POPULATIONS

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Abstract. We study dynamic markets in which participants are randomly matched to bargain over the price of a heterogeneous good. There is a continuum of players drawn from a finite set of types. Players exogenously enter the market over time and then exit upon trading. At every date, the matching probabilities for each pair of types are endogenously determined by the distribution of players in the market. A player's bargaining power at any stage depends on intra- and inter-temporal variations in the potential gains from trade, the feasible agreements at future dates, and the induced distribution of bargaining partners. We establish that an equilibrium always exists. Moreover, all equilibria that feature the same evolution of the macroeconomic variables are payoff equivalent. However, we show that multiple self-fulfilling expectations about the trajectory of the economy, generating distinct equilibrium dynamics and payoffs, may coexist. We also prove the existence of steady states in stationary environments. Our analysis extends and complements several models of bargaining in markets.

Keywords: bargaining, decentralized, dynamic markets, random matching, heterogenous goods, equilibrium existence, multiplicity, iterated conditional dominance, steady states.

1. Introduction

We study decentralized dynamic markets in which agents bargain over the price of a heterogeneous good. The surplus that pairs of market participants can generate from trade may differ due to variations in valuations or good quality; cost of transportation between various locations; trade laws (tariffs, trade barriers, quality standards for imports); productivity and disutility of labor. The availability and size of the surplus may also depend on the strength of social relationships, business connections, and exposure to various advertising platforms.

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Product features that are relevant to customers also lead to match specific values. For instance, buyers of used cars care about the vehicle's make, mileage, manufacturing year, fuel efficiency, and so on. In the market for apartment rentals, search is typically driven by location, the number of bedrooms, and the state of the appliances.

The distribution of bargaining opportunities that market participants face may change over time. The stock of potential trading partners and the amount of surplus available at any date depend on the inflows of agents into the market, the matching frequencies, as well as the outflows of agents who complete transactions. The participants need to forecast the evolution of the macroeconomy, as determined by the endogenous volume of trade and the relative matching probabilities induced by inflows and outflows, and negotiations should reflect the anticipated market conditions.

We analyze such decentralized markets in the context of an infinite horizon bargaining game played in discrete time. The set of player types is finite, and there is a continuum of players of each type. Players exogenously enter the game over time and leave only upon reaching an agreement. A fraction of the players is matched to bargain in every period. The measure of players of types i and j matched to bargain with one another at a particular date depends on the type distribution in the market at that date. Every player is involved in at most one match at a time. For each matched pair of types i and j, one of the two players is chosen to make an offer to the other specifying a division of the surplus s_{ij} between the two of them. If the offer is accepted, the two players exit the game with the shares agreed on. If the offer is rejected, the players stay in the game for the next period. The players of any given type have a common discount factor.

Our setting encompasses a number of models from the literature on bargaining in markets.[1] The two-type case, in which pairs of players of the same type cannot generate surplus, effectively corresponds to the influential model of Rubinstein and Wolinsky (1985). Binmore and Herrero (1988 a,b) developed the study of the two-type case in non-stationary environments. The market of buyers and sellers with heterogenous reservation values analyzed by Gale (1987) can be obtained by setting $s_{ij} = \max(0, v_i - v_j)$ for buyer-seller pairs (i, j), where v_k denotes the valuation of agent k for the good, and $s_{ij} = 0$ whenever i and j are either both buyers or both sellers. The model of Manea (2011), in which a network represents the

[1] Osborne and Rubinstein (1990) provide a survey of the early theoretical research on bargaining in markets.

pattern of interaction or compatibility between player types and links have identical values, corresponds to the situation in which s_{ij} equals 1 if i and j are linked in the network and 0 otherwise.

As the opening remarks suggest, the structure of equilibria in our dynamic setting entails a complex relationship between several objects of infinite dimension. A player's payoff at any point in time incorporates the surplus heterogeneity within and across periods, the bargaining power of his partners, and the feasible agreements at future dates. The incentives for agreements depend in turn on the distribution of player types at every stage and the induced path of matching frequencies. Nevertheless, we establish the existence of an equilibrium. The proof technique may be useful in other dynamic environments. We note parenthetically that the result complements the analysis of Gale (1987). The latter paper explores properties of equilibria abstracting away from existence issues.

We establish a payoff equivalence result for equilibria that generate the same path of market distributions. Restrict attention to equilibria in which no (infinitesimal) player can affect the macroeconomic variables by unilaterally changing his strategy. In such equilibria players take the matching probabilities along the equilibrium path as given. Thus on-path incentives in the benchmark bargaining game are equivalent to those in an alternative model where the matching probabilities are exogenously specified. We show that the latter model can essentially be solved using iterated conditional dominance. Hence all equilibria of the model with exogenous matching probabilities are payoff equivalent. This conclusion generalizes uniqueness results from Binmore and Herrero (1988b) and Manea (2011). We develop a procedure to compute the unique equilibrium payoffs with any degree of accuracy.

Thus the model with exogenous matching probabilities provides a partial equilibrium approach to predicting payoffs for a given evolution of the macroeconomy. The properties of the unique payoffs compatible with a postulated market path play a key role in the proof of equilibrium existence for the benchmark model. Notwithstanding, the alternative model can also be interpreted as a free-standing depiction of situations in which players have stubborn beliefs about the evolution of the macroeconomy.

We show that the benchmark model does not necessarily have a unique equilibrium. Indeed, we produce an example that accommodates multiple consistent theories about relative bargaining power, feasible agreements, and the trajectory of the economy. We interpret the possibility of multiple equilibria as a manifestation of market sentiment. Expectations about future market developments play a crucial role in the dynamic of negotiations and can act as self-fullfilling prophecies.

Rubinstein and Wolinsky (1985), Gale (1987), and Manea (2011) consider stationary bargaining games in which players who reach agreement are replaced by identical, new players in the next round. Their characterizations of equilibrium outcomes are contingent on the economy being at a steady state. It is natural to ask how the distribution of player types is determined in the steady state of an economy with an exogenous set of potential market entrants. The stationary bargaining games of the aforementioned papers can be interpreted as special instances of the model with exogenous matching probabilities. The characterization of equilibrium behavior in the latter model can be used to understand the mechanics of steady states. Specifically, we assume that the inflows and matching technology are time independent, and we investigate the existence of a stationary market compatible with the inflows.

Suppose that every player type incurs a small cost to enter the market. Then entry decisions depend on the costs of entry and the payoffs in the bargaining game. In a steady state, the inflows balance the endogenous equilibrium outflows of players who trade in the game. We establish that if the matching process satisfies a mild regularity condition, then a steady state exists for every configuration of small entry costs. An alternative existence result shows that for any continuous matching process, one can set arbitrarily low entry fees such that the resulting economy has a steady state in which each population has positive size.

The rest of the paper is organized as follows. The next section defines the benchmark model and establishes equilibrium existence. In Section 3 we introduce the model with exogenous matching probabilities and show that it is conditional dominance solvable. We discuss equilibrium multiplicity for the benchmark bargaining game in Section 4. The existence of steady states is addressed by Section 5. Section 6 concludes and the Appendix contains the proofs.

2. The Benchmark Model

We consider dynamic markets with a set $N = \{1, 2, \ldots, n\}$ of **populations** or **player types**. A pair of players from populations i and j can generate a **surplus** $s_{ij} = s_{ji} \geq 0$. In every period $t = 0, 1, \ldots$, an endogenously determined measure $\mu_{it} \geq 0$ of players i participates in the market. It is always the case that $\sum_{i \in N} \mu_{it} > 0$. Hence the **market state** at time t is described by a vector of population sizes $\mu_t = (\mu_{it})_{i \in N} \in [0, \infty)^n \setminus \{\mathbf{0}\}$ ($\mathbf{0}$ denotes the vector with n zero components). For every market state μ_t at time t there is a **matching technology** such that, for all $(i, j) \in N \times N$, a measure $\beta_{ijt}(\mu_t) \geq 0$ of players i has the opportunity to make an offer to one of the players j.[2] [3] The function β_{ijt} is continuous on $[0, \infty)^n \setminus \{\mathbf{0}\}$. No player is involved in more than one match (as either proposer or responder) at a time, so

$$\mu_{it} \geq \sum_{j \in N} \beta_{ijt}(\mu_t) + \beta_{jit}(\mu_t), \forall i \in N. \tag{2.1}$$

We assume that a positive measure of players is left unmatched every period, that is, for every t and μ_t there exists a population i for which the inequality above is strict.

The matching technology treats all players of the same type symmetrically in the following sense. Each player of type i is equally likely to be one of the $\beta_{ijt}(\mu_t)$ players i given the opportunity to make an offer to players j in period t. Thus a player of type i is selected to make an offer to one of type j with **probability**[4]

$$\pi_{ijt}(\mu_t) = \lim_{\substack{\tilde{\mu}_t \to \mu_t \\ \tilde{\mu}_{it} > 0}} \frac{\beta_{ijt}(\tilde{\mu}_t)}{\tilde{\mu}_{it}}. \tag{2.2}$$

For $\mu_{it} > 0$, the continuity of β_{ijt} implies that the limit above is well-defined and is simply given by $\beta_{ijt}(\mu_t)/\mu_{it}$. We assume that the limit also exists for all $\mu_t \in [0, \infty)^n \setminus \{\mathbf{0}\}$ with $\mu_{it} = 0$. It follows that the function π_{ijt} is continuous on $[0, \infty)^n \setminus \{\mathbf{0}\}$. We do not explicitly

[2] We allow for the possibility that players from the same population i are matched to one another, that is, $\beta_{iit}(\mu_t) > 0$.

[3] In our setting, the conditions $\beta_{ijt}(\mu_t) = 0, \forall t, \mu_t$ and $s_{ij} = 0$ are equivalent. For instance, Manea (2011) assumes that any two players who are not linked in a network are never matched to bargain, which can be alternatively interpreted as the inability of disconnected pairs of players to generate surplus.

[4] The probability that a player i is selected to receive an offer from any player j in period t can be defined analogously, but is inconsequential to our analysis.

model the matching process since the functions π_{ijt} constitute a sufficient statistic for our analysis.[5]

A salient class of matching technologies that satisfy our continuity requirements is obtained by assuming that every player i meets another player with a fixed probability p, and the conditional probability of i meeting a type j is proportional to the size of population j. Each player of type i is recognized as a proposer in half of the matched pairs (i, j). The corresponding matching technology is described by

$$\begin{aligned} \beta_{ijt}(\mu_t) &= \frac{p}{2}\frac{\mu_{it}\mu_{jt}}{\sum_{k\in N}\mu_{kt}} \\ \pi_{ijt}(\mu_t) &= \frac{p}{2}\frac{\mu_{jt}}{\sum_{k\in N}\mu_{kt}}, \forall i, j \in N, t \geq 0, \mu_t \in [0,\infty)^n \setminus \{\mathbf{0}\}. \end{aligned} \tag{2.3}$$

A similar definition appears in Gale (1987). While it may be helpful to interpret the results in the context of such simple matching technologies, it should be emphasized that our analysis applies generally.

The **benchmark bargaining game** is as follows. A measure $\lambda_{i0} \geq 0$ of players of type i is initially present in the game. We assume that $\sum_{i\in N}\lambda_{i0} > 0$ and also use the notation $\mu_0 = \lambda_0$. Every period $t = 0, 1, \ldots$, players are randomly matched to bargain according to $\beta_t(\mu_t)$. A player i chosen to make an offer to some player j can either propose a division of the surplus s_{ij} or decline to bargain with j. If i makes an offer that j accepts, then the two players exit the game with the shares agreed upon. If i makes an offer that j rejects or i declines to bargain with j, then the two players remain in the game for the next period. In period $t+1$, a measure $\lambda_{i(t+1)} \geq 0$ of new players i enters the market, joining the players from earlier stages who have not yet reached an agreement. The total stock of players i at

[5]We can construct a matching procedure that generates the desired matching probabilities for populations of positive measure by adapting "the roulette method" of Alos-Ferrer (1999). Suppose that the set of players i participating in the market at time t is transformed into the interval $[0, \mu_{it})$ through a measure-preserving map. Then each player of type i is identified with some $\tilde{i} \in [0, \mu_{it})$. For every $i \in N$, let $f_i : [0, \mu_{it}) \to \cup_{j\in N}\{(i, j), (j, i)\} \cup \{0\}$ be an arbitrarily measurable function such that the Borel measures of the pre-images of (i, j) and (j, i) for $j \neq i$ are $\beta_{ijt}(\mu_t)$ and $\beta_{jit}(\mu_t)$, respectively, and the measure of the pre-image of (i, i) is $2\beta_{iit}(\mu_t)$. Let $(x_i)_{i\in N}$ be a collection of independent random variables, with x_i uniformly distributed over $[0, \mu_{it})$. For every realization $(\tilde{x}_i)_{i\in N}$ of the random variables, for all $i \neq j$, the sets of players $\tilde{i} \in [0, \mu_{it})$ and $\tilde{j} \in [0, \mu_{jt})$ satisfying

$$f_i((\tilde{i} + \tilde{x}_i) \bmod \mu_{it}) = (i, j) = f_j((\tilde{j} + \tilde{x}_j) \bmod \mu_{jt})$$

both have measure $\beta_{ijt}(\mu_t)$ (for $b > 0$, we use the notation $a \bmod b$ for the unique $c \in [0, b)$ such that $(a-c)/b$ is an integer). Then there exists a measure-preserving bijection from the former set to the latter, which we use to match players of types i and j, with players i in the role of the proposer. Similarly, we can match the mass of $2\beta_{iit}(\mu_t)$ players $\tilde{i}$ satisfying $f_i((\tilde{i} + \tilde{x}_i) \bmod \mu_{it}) = (i, i)$ with one another.

the beginning of period $t+1$ is denoted by $\mu_{i(t+1)}$.[6] The players of type i have a common discount factor $\delta_i \in (0,1)$.[7]

The model is flexible in terms of the amount of information each player has about other players' past matchings and outcomes. One possible treatment assumes perfect information, which entails that all players observe the entire history of realized matchings and ensuing negotiations. Alternatively, players may have partial knowledge of others' past bargaining encounters—e.g., each player observes only the outcomes of his own interactions; additionally, players may be aware of the realized matches, but not of the details of each negotiation; or players learn only about the experiences of their own population. However, we retain the key assumptions that all players observe the size of each population at the beginning of every period and that matched players know each other's type.

In the case of perfect information, the solution concept we use is that of subgame perfect equilibrium. For versions of the game with imperfect information, we introduce the concept of **belief-independent equilibrium**. A strategy profile constitutes a belief-independent equilibrium for an extensive form game if every player's strategy is optimal conditional on each information set against all beliefs at that information set. In other words, a strategy profile is a belief-independent equilibrium if it is sequentially rational with respect to every profile of beliefs.[8] For either solution concept, we restrict attention to equilibria that are **robust** in the sense that no (infinitesimal) player can affect the population sizes along the path by changing his own strategy. Our results apply for all types of information structure discussed above, and henceforth we simply refer to the corresponding solution concept as equilibrium.[9]

Several technical assumptions are necessary to guarantee that the stock of players present in the market at every stage is measurable. The set of players i present in the game at time

[6]The condition $\sum_{i \in N} \lambda_{i0} > 0$, along with the assumption that a positive measure of players is left unmatched every period, implies that $\mu_t \neq \mathbf{0}$ for all $t \geq 0$.

[7]A player who never reaches an agreement obtains a zero payoff.

[8]Note that in our setting each information set includes knowledge of the current market distribution, so that all agents can correctly asses the matching probabilities.

[9]The equilibria we construct have a strong Markovian flavor in that at any point in time each player's behavior depends only on the underlying market distribution and the type of his bargaining partner. However, formalizing the idea of payoff relevant histories (Maskin and Tirole 2001) for imperfect information versions of our game—particularly for cases in which players are uncertain about the realized market-wide matching at some stages—is beyond the scope of this paper.

t can be transformed into the interval $[0, \mu_{it})$ through a measure-preserving bijection. The matching process is a probability distribution over proposer-responder assignments (which leaves out some players) with the property that a measurable set of players i is selected to make (receive) an offer to (from) one of the players j. It is only possible to formally define strategy profiles under which the set of players from each population reaching agreement at any history is measurable. We also need to restrict attention to pure strategies (see Aumann (1964)). Note that the macroeconomic effects of mixing can be replicated by the idea of distributional strategies (Milgrom and Weber 1985).

Note that players drawn from populations of measure zero may be matched for bargaining with positive probability and enjoy positive payoffs. However, the existence of such players does not directly impact the matching probabilities and the expected payoffs of other market participants. Allowing the size of a population to vanish at some date may seem pedantic at this stage, but will become useful for the analysis of steady states in Section 5. Up to that point, the reader may focus on the case in which the inequalities 2.1 hold strictly and $\lambda_{i0} > 0$ for all $i \in N$. The latter conditions guarantee that no population is ever completely depleted.

The existence of an equilibrium in our dynamic setting is not straightforward because the rate of departures following agreements is endogenously determined in equilibrium, and the matching probabilities depend in turn on the population sizes. There is a complex relationship between the evolution of the payoffs for each population of players and the path of feasible agreements. Our first result establishes equilibrium existence. The proof of this and subsequent results may be found in the Appendix.

Theorem 1. *An equilibrium exists for the bargaining game.*

In Section 4 we show that the equilibrium is not necessarily unique. However, a partial uniqueness result holds for robust equilibria that lead to the same path of market distributions. More generally, payoff equivalence is obtained in an environment where the path of matching probabilities is exogenously given. The latter model, which we formally introduce in the next section, can be used to describe behavior on the equilibrium path in the benchmark model—in particular, it provides a building block for the proof of Theorem 1—but is also of independent interest.

To sketch the proof of Theorem 1, define the spaces of paths of agreement rates, market distributions, matching probabilities, and feasible payoffs, respectively, as follows

$$
\begin{aligned}
\mathcal{A} &= \{(a_{ijt})_{i,j\in N,t\geq 0} | a_{ijt} \in [0,1], \forall i,j \in N, t \geq 0\} \\
\mathcal{M} &= \{(\mu_{it})_{i\in N,t\geq 0} | \mu_0 = \lambda_0; \mu_{it} \in [0, \sum_{\tau=0}^{t} \lambda_{i\tau}], \forall i \in N, t \geq 1\} \\
\mathcal{P} &= \{(p_{ijt})_{i,j\in N,t\geq 0} | p_{ijt} \in [0,1], \forall i,j \in N, t \geq 0\} \\
\mathcal{V} &= \{(v_{it})_{i\in N,t\geq 0} | v_{it} \in [0, \max_{j\in N} s_{ij}], \forall i \in N, t \geq 0\}.
\end{aligned}
$$

We construct a correspondence $f : \mathcal{A} \rightrightarrows \mathcal{A}$ by composing the correspondence α and the functions v^*, π, κ, where

$$
\mathcal{A} \xrightarrow{\kappa} \mathcal{M} \xrightarrow{\pi} \mathcal{P} \xrightarrow{v^*} \mathcal{V} \overset{\alpha}{\rightrightarrows} \mathcal{A}.
$$

The latter maps are specified as follows

- $\kappa(a)$ describes the path of the market under the assumption that a fraction a_{ijt} of the proposer-responder pairs (i,j) matched at time t reaches agreement
- $\pi(\mu)$ denotes the matching probabilities along the market path μ, as specified by 2.2 (with a minor abuse of notation)
- $v^*(p)$ represents the unique equilibrium payoffs in the model with exogenous matching probabilities p (characterized by Theorem 2 in Section 3)
- $\alpha_{ijt}(v)$ defines the set of agreement rates that are incentive compatible for proposer-responder pairs (i,j) matched at time t, assuming that bargaining proceeds as if the expected disagreement payoffs at $t+1$ were given by v_{t+1}.

Note that while κ and π stem from the physical constraints of the environment, v^* and α reflect (hypothetical) equilibrium conditions.

We can verify that $f = \alpha \circ v^* \circ \pi \circ \kappa$ satisfies the hypotheses of the Kakutani-Fan-Glicksberg theorem and thus it has a fixed point a^*. We then construct a robust equilibrium in which agreements arise at the rates described by a^*, the market follows the path $\kappa(a^*)$, and the payoffs are given by $v^*(\pi(\kappa(a^*)))$. At stages where the trajectory of the economy diverges from $\kappa(a^*)$, strategies are derived from a fixed point of an appropriately modified correspondence (the set $\mathcal{M}$ and the function a need to be redefined taking into account the state of the market at the first stage where divergence occurs). Departures from the path of market distributions induced by the latter strategies are treated analogously, and so on.

The sketch of the proof above provides intuition about how different elements of the game fit together. In particular, it highlights the relationship between payoffs and matching probabilities, which we further explore in the next section.

Remark 1. The structure of agreements may seem an unusual starting point for our fixed point construction. Paths of payoffs and market distributions constitute more natural primitives for describing equilibrium outcomes. These variables suggest the study of the following map compositions

$$\mathcal{V} \overset{\alpha}{\rightrightarrows} \mathcal{A} \overset{\kappa}{\to} \mathcal{M} \overset{\pi}{\to} \mathcal{P} \overset{v^*}{\to} \mathcal{V}$$

$$\mathcal{M} \overset{\pi}{\to} \mathcal{P} \overset{v^*}{\to} \mathcal{V} \overset{\alpha}{\rightrightarrows} \mathcal{A} \overset{\kappa}{\to} \mathcal{M}.$$

However, neither of the compositions above is necessarily convex valued due to the nonliniarities in κ induced by the general matching process. Then standard fixed point theorems are not applicable.

Remark 2. A more traditional approach to proving the existence of equilibria in dynamic games does not seem tractable in our setting. The bargaining game can be approximated by a sequence of finite horizon truncations as in Fundenberg and Levine (1983). Limit points of robust equilibria of the truncated games constitute equilibria of the infinite horizon game. In order to establish the existence of an equilibrium for the benchmark bargaining game, it suffices to show that every finite period version admits a robust equilibrium. The latter step typically involves an inductive argument on the length of the game (similar to backward induction). Consider a finite horizon version of the bargaining game and suppose that an equilibrium exists for all shorter games. In an equilibrium of the game under consideration, the second period play and payoffs depend on the first period volume of trade. We are led to define a correspondence that maps first period agreement rates into continuation equilibrium payoffs. Each profile of second period payoffs in turn determines what first period agreements are incentive compatible. Note that both of the outlined correspondences may be set valued. Fixed points of the composition of the two correspondences describe first period trade and second period payoffs in equilibria of the considered game. The two correspondences are upper hemicontinuous, and so is their composition. However, it seems theoretically plausible that the composition (in either order) is not convex valued (even if we convexify the former correspondence by the introduction of a public randomization device). Thus classic fixed point results do not directly apply. It is also unclear whether the problem may be circumvented by showing that the former correspondence admits a continuous selection that we can exploit in the construction instead. Dutta and Sundaram (1998) discuss related issues with using backward induction to prove the existence of Markov perfect equilibria in finite horizon stochastic games.

3. An Alternative Model

We consider the following **model with exogenous matching probabilities**. Players from the n populations are present in the market in every period $t = 0, 1, \ldots$ We are agnostic about the composition of the market at each date. We assume that every player of type i is given the opportunity to make an offer to one of the players j in period t with the exogenous probability $p_{ijt} \geq 0$.[10] [11] A player remains in the market until he reaches an agreement. Payoffs are specified as in the benchmark model.

The sketch of the game above is purposely vague regarding the set of new players entering the market every period, the exact matching procedure, and the information structure. It is conceivable that knowledge of the latter elements could permit players to make complex inferences about the state of the market. However, the nature of these inferences does not significantly affect the equilibrium outcomes of the *class of games* sharing the qualities outlined above. We are able to make sharp predictions about equilibrium behavior without keeping track of the inflows and outflows for each population, the details of the matching procedure, and the beliefs every player holds. Indeed, we will show that the probabilities (p_{ijt}) completely characterize the strategic situation for all players.

Technically, one can imagine that the matching probabilities are held fixed under the matching technology from the benchmark model by adjusting the inflows into the market in response to the outflows of agents reaching agreement every period. As suggested earlier, the analysis of the model with exogenous matching probabilities can be alternatively

[10]We maintain the assumption that every matched player knows the type of his partner.

[11]The probability that i receives an offer from j will be irrelevant to the analysis.

regarded as a partial equilibrium approach to predicting payoffs for a certain evolution of the macroeconomic conditions over time.

As a free-standing piece, the model describes a market with behavioral participants. Players start with identical beliefs about the path of matching probabilities and never revise these expectations in response to information they receive. This is reasonable in a setting where agents rely on public predictions of the macroeconomic variables and ignore evidence that seems inconsistent with the projections. In a large market where mistakes are possible, agents may think that their own past interactions and observations do not necessarily reflect future trends.

In the strategic environment of the exogenous matching probabilities model, we establish that all belief-independent equilibria are payoff equivalent. We actually prove a stronger claim: behavior is essentially pinned down by a process of iterated conditional dominance analogous to that proposed by Fudenberg and Tirole (1991, Section 4.6) in the context of multi-stage games with observed actions. In our setting, an action a available at an information set h of some player i is **conditionally dominated** if, for every belief ν over the decision nodes at h, any strategy of i that assigns positive probability to a is strictly dominated by another strategy when i's payoffs are evaluated with respect to the information set h and the beliefs ν. **Iterated conditional dominance** is the process that, at every stage, eliminates all conditionally dominated actions at each information set, given the opponents' strategies that survived the earlier stages of elimination. The following result summarizes properties of the equilibrium behavior and the unique equilibrium payoffs.

Theorem 2. *There exists a vector of payoffs $(v^*_{it}(p))_{i\in N,t\geq 0}$ such that every bargaining game embedded in the model with exogenous matching probabilities p satisfies the following properties.*

(i) The only period t actions that may survive iterated conditional dominance specify that player i reject any offer smaller than $\delta_i v^*_{i(t+1)}(p)$ and accept any offer greater than $\delta_i v^*_{i(t+1)}(p)$.
(ii) A belief-independent equilibrium exists.
(iii) In every belief-independent equilibrium, the expected payoff of any player i present at the beginning of period t is $v^*_{it}(p)$.

(iv) The equilibrium payoffs $(v^*_{it}(p))_{i\in N,t\geq 0}$ *constitute the unique bounded solution* $(v_{it})_{i\in N,t\geq 0}$ *to the system of equations*

$$v_{it} = \sum_{j\in N} p_{ijt} \max\left(s_{ij} - \delta_j v_{j(t+1)}, \delta_i v_{i(t+1)}\right) + \left(1 - \sum_{j\in N} p_{ijt}\right) \delta_i v_{i(t+1)}. \tag{3.1}$$

(v) The payoffs $v^*_{it}(p)$ *vary continuously in* p *for all* $i \in N, t \geq 0$.

Corollary 1. *All equilibria of the benchmark bargaining game that lead to the same path of population distributions are payoff equivalent.*

The proof of Theorem 2 can be easily adapted to show uniqueness of the security equilibrium payoffs for the model with exogenous matching probabilities. The latter equilibrium concept has been introduced by Binmore and Herrero (1988b).[12] The alternative statement of Theorem 2 asserting payoff equivalence of security equilibria generalizes Theorem 6.3 of Binmore and Herrero (1988b) to settings with more than two populations. In turn, the latter result represents an extension of the analysis of Rubinstein and Wolinsky (1985) to non-stationary environments.

The derivation of bounds for the offers and payoffs that survive iterated conditional dominance rely on implicit conjectures about which matches lead to trade. The bounds need to be tight in order to yield unique payoffs, so they must reflect precise estimates of the best and worst case scenarios for every player and each potential bargaining partner. In particular, it is not a priori clear whether the best and worst case scenarios for a given match involve an agreement.

In general, solving the infinite system of equations 3.1 that characterizes the equilibrium payoffs may be intractable. Nonetheless, we can implement the following computational procedure to estimate the equilibrium payoffs. Define the sequences $(m^k_{it})_{i\in N,t\geq 0}$ and $(M^k_{it})_{i\in N,t\geq 0}$

[12]The relationship between iterated conditional dominance and security equilibrium has not been established for general dynamic games.

recursively for $k = 0, 1, \ldots$ as follows

$$m_{it}^0 = 0, M_{it}^0 = \max_{j \in N} s_{ij}$$

$$m_{it}^{k+1} = \sum_{j \in N} p_{ijt} \max\left(s_{ij} - \delta_j M_{j(t+1)}^k, \delta_i m_{i(t+1)}^k\right) + \left(1 - \sum_{j \in N} p_{ijt}\right) \delta_i m_{i(t+1)}^k$$

$$M_{it}^{k+1} = \sum_{j \in N} p_{ijt} \max\left(s_{ij} - \delta_j m_{j(t+1)}^k, \delta_i M_{i(t+1)}^k\right) + \left(1 - \sum_{j \in N} p_{ijt}\right) \delta_i M_{i(t+1)}^k.$$

The proof of Theorem 2 establishes that, for all $k \geq 0$, under the strategies that survive iterated conditional dominance, every player of type i rejects any offer smaller than $\delta_i m_{i(t+1)}^k$ and accepts any offer greater than $\delta_i M_{i(t+1)}^k$ in period t (regardless of the identity of the proposer). Both sequences $(m_{it}^k)_{k \geq 0}$ and $(M_{it}^k)_{k \geq 0}$ converge to $v_{it}^*(p)$ as $k \to \infty$, and $v_{it}^*(p) \in [m_{it}^k, M_{it}^k]$ for all $k \geq 0$. We also show that for every $i \in N, t \geq 0, k \geq 0$,

$$0 \leq M_{it}^k - m_{it}^k \leq (\max_{j \in N} \delta_j)^k \max_{j, j' \in N} s_{jj'}.$$

Therefore, the equilibrium payoffs $v_{i0}^*(p)$ can be approximated by the interval $[m_{i0}^k, M_{i0}^k]$, the length of which declines exponentially in k. Note that the number of steps required to compute m_{i0}^k and M_{i0}^k is linear in k.

4. Equilibrium Multiplicity

In this section we analyze the structure of the equilibria of the bargaining game in a two-population setting ($n = 2$). We identify a range of parameters for which multiple equilibria exist. Assume that $s_{11} = a \in (1, 2], s_{12} = s_{22} = 1$ and $\delta_1 = \delta_2 = \delta \in [0, 1)$. Suppose that the initial market distribution is given by $\mu_{10} = x \in [1/2, 1), \mu_{20} = 1 - x$ and that no new players enter the economy after the first period ($\lambda_{it} = 0$ for all $t \geq 1$).

Players are matched to bargain following the protocol from 2.3 with $p = 1/2$. Thus the probability that a player i is selected to make an offer to some player j in the period t market μ_t is

$$\pi_{ijt}(\mu_t) = \frac{\mu_{jt}}{4(\mu_{1t} + \mu_{2t})}.$$

Hence the proportion of players of type 1 present in the market, $\mu_{1t}/(\mu_{1t} + \mu_{2t})$, constitutes a sufficient statistic for the matching probabilities at time t. We refer to the latter ratio as the **index** of the market μ_t.

We inquire into the existence of two types of equilibria. In an **all-agreement equilibrium** all matches along the equilibrium path result in agreement. A **population-agreement equilibrium** has the property that on the equilibrium path, in every period, each player reaches agreements when matched to bargain with players from his own population, but not from the other. Each type of equilibrium leads to a particular path of market distributions and generates unique payoffs in light of Corollary 1. We can evaluate the **total welfare** of an equilibrium as x times the expected payoff of a player of type 1 in that equilibrium plus $1-x$ times the payoff of a player 2.

Note that, under the assumed matching technology, if all pairs of players matched at time t in the market μ_t reach agreement, then the next period market distribution is $\mu_{t+1} = \mu_t/2$. Then on the equilibrium path of the all-agreement equilibrium the market index must be x in every period, and the unique payoffs of players from the same population are stationary. This makes the computation of payoffs in the candidate equilibrium straightforward. The formulae for payoffs in the population-agreement equilibrium are not as tractable. If agreements arise as postulated in the latter equilibrium, play proceeds from a market with index y to one with index $y(2-y)/(1+2y(1-y))$. In particular, the market index declines over time. The non-trivial evolution of market indices complicates the estimation of the ranges of parameters where the two types of equilibria (co)exist and the comparison of payoffs across equilibria.

Proposition 1 below shows that the two types of equilibria coexist for a non-generic range of parameters. When both equilibria exist, players of type 1 are better off in the population-agreement equilibrium, while players 2 prefer the all-agreement one. However, the two types of equilibria cannot be consistently ranked according to their welfare.

Proposition 1. *Fix $a \in (1,2]$.*

(i) For every $x \in [1/2, 1)$, there exist $\bar{\delta}(x)$ and $\underline{\delta}(x)$ such that an all-agreement equilibrium exists if and only if $\delta \leq \bar{\delta}(x)$, and a population-agreement equilibrium exists if and only if $\delta \geq \underline{\delta}(x)$.

(ii) If $x \in ((a+1)/4, 1)$, then $\bar{\delta}(x) > \underline{\delta}(x)$, and both equilibria exist for $\delta \in [\underline{\delta}(x), \bar{\delta}(x)]$.

(iii) For every profile of parameters for which both types of equilibria exist, the payoff of a player of type 1 in the all-agreement equilibrium is not greater than that in the

population-agreement one. Players of type 2 have the opposite (weak) preferences over the two equilibria.

(iv) *The two types of equilibria are not consistently ranked in terms of total welfare: for every $a \in (1, 4/3)$, there exists $\varepsilon > 0$ such that the all-agreement equilibrium yields higher welfare than the population-agreement equilibrium for $x \in ((a+1)/4, (a+1)/4+\varepsilon)$ and $\delta = \bar{\delta}(x)$, and the comparison is reversed for $x \in (1-\varepsilon, 1)$ and $\delta = \bar{\delta}(x)$.*

To gain some intuition into the coexistence of the two equilibria, note first that the players of type 1 are intrinsically more powerful because they can generate a surplus $a > 1$ when matched to bargain with one another, while the other pairs of types create only a unit surplus. Moreover, players 1 are given the opportunity to realize the surplus a frequently since population 1 constitutes a proportion $x > 1/2$ of the total mass of market participants. By the same token, the players of type 2 are more likely to be matched with players from population 1 than with other players 2. All matches involving population 2 generate one unit of surplus, but players 1 are relatively stronger than players 2, so the players of type 2 often encounter unfavorable partners. Thus the matching process further boosts the bargaining power of players 1 and undermines the position of players 2. We refer to the impact of the greater amount of surplus available within population 1 on relative bargaining strengths as the *surplus effect*, and to the ramifications of this effect, amplified by the larger size of population 1 via the matching probabilities, as the *frequency effect*.

Consider now an all-agreement equilibrium. As explained earlier, the market index is constant along the equilibrium path. The players from population 2 allow the frequency effect to propagate over time by trading with players of type 1. In effect, the players 1 exploit the self-inflicted weakness of players 2. The dynamic is different in the context of a population-agreement equilibrium. By refusing to trade with players from population 1, the players 2 secure a market path with declining indices and diminishing frequency effect. The bargaining position of players 2 steadily improves over time, and the prospect of higher future payoffs provides incentives for them to avoid agreements with population 1. Therefore, the divergence of the two market paths leads to differences in the magnitude of the frequency effect, which create a wedge in the relative bargaining power of the two populations that overturns the incentives for across population trade.

The two equilibria embody contrasting expressions of *market sentiment*. On the one hand, in the all-agreement equilibrium players 2 hold the pessimistic beliefs that all matchings between populations 1 and 2 result in agreement. A persistent frequency effect is expected to emerge. On the other hand, in the population-agreement equilibrium players 2 optimistically anticipate that populations 1 and 2 do not engage in trade with one another. The frequency effect is expected to gradually decline. In both cases, the predicted trajectory of the economy becomes a self-fulfilling prophecy: the anticipated agreements are incentive compatible.

Remark 3. The analysis of this section is reminiscent of the multiplicity of steady states in a two-type example from the context of the search model of Burdett and Coles (1997, 1999). It is important to clarify the differences. Burdett and Coles fix some stationary inflows and restrict attention to steady states (in an exercise analogous to Section 5 below). The initial market composition is endogenously determined in their model. The two types of equilibria constructed by Burdett and Coles start with distinct market compositions and induce constant paths of market indices. By contrast, we allow for non-stationary dynamics in a setting where both the initial market distribution and the future inflows are exogenous. The paths of the market index in our equilibria originate from the same point and gradually diverge. In particular, the population-agreement equilibrium features a declining path of market indices.

5. Steady States

This section focuses on stationary environments. Specifically, we assume that the inflows and the matching process are time independent. In this context, we explore the properties of steady states, which are defined by equilibria of the bargaining game that lead to constant population sizes over time. We modify the previous notation as follows: $\lambda_i > 0$ is the measure of new players of type i in every period, $\mu = (\mu_i)_{i \in N} \in [0, \infty)^n \setminus \{\mathbf{0}\}$ describes the size of each population in a potential steady state, $\beta_{ij}(\mu)$ and $\pi_{ij}(\mu)$ represent the measure of players i who can make an offer to some player j at any date where the market distribution is μ and the probability that a given player of type i is involved in such a match, respectively. We retain the assumptions that guarantee the continuity of π_{ij} over $[0, \infty)^n \setminus \{\mathbf{0}\}$ for all $i, j \in N$.

In a robust equilibrium with a market distribution $\mu \in [0, \infty)^n \setminus \{\mathbf{0}\}$ at every date, incentives on the path map to the equilibrium conditions in the model with exogenous matching probabilities where $p_{ijt} = \pi_{ij}(\mu)$ for all $t \geq 0$. We denote the unique equilibrium payoffs of the corresponding class of games by $v(\mu)$. Theorem 2 establishes that $v(\mu)$ is the only bounded solution $(v_{it})_{i \in N, t \geq 0}$ to the system of equations 3.1. However, since $p_{ijt} = p_{ij(t+1)}$ for all $i, j \in N, t \geq 0$, the one-period forward translation v' of $v(\mu)$ (defined by $v'_{it} = v_{i(t+1)}(\mu)$ for $i \in N, t \geq 0$) also constitutes a bounded solution for that system, thus $v_{it}(\mu) = v'_{it} = v_{i(t+1)}(\mu)$ for all $i \in N, t \geq 0$. This shows that the equilibrium payoffs in a steady state are constant over time. We simply use the notation $v_i(\mu)$ for the common expected payoff of all players of type i at any date. Hence $v(\mu)$ solves[13]

$$v_i(\mu) = \sum_{j \in N} \pi_{ij}(\mu) \max\left(s_{ij} - \delta_j v_j(\mu), \delta_i v_i(\mu)\right) + \left(1 - \sum_{j \in N} \pi_{ij}(\mu)\right) \delta_i v_i(\mu), \forall i \in N.$$

We first argue that even with minimal equilibrium restrictions on the set of agreements, regardless of the matching technology, it may be that no feasible mass of pairwise departures from the market perfectly balances the inflows.

Example 1. Suppose there are two populations with $s_{11} = s_{22} = 0, s_{12} > 0$. Then in every equilibrium agreements are reached only between players of distinct types. Hence any matching process leads to equal measures of players 1 and 2 exiting the market every period. If $\lambda_1 \neq \lambda_2$ then it is impossible for inflows of λ_1 players 1 and λ_2 players 2 to match the outflow of pairs of players 1 and 2.

We next argue that introducing small entry costs may stabilize the market. In the setting of Example 1, suppose that measures λ_1 of players 1 and λ_2 of players 2 enter the game every period, with $\lambda_1 > \lambda_2$. Assume additionally that a measure λ_2 of players 2 exits the market per period. Then the population of players 1 grows without bound over time, while the size of population 2 remains constant. This means that the probability a player 1 gets matched to bargain in any given period becomes vanishingly small. Hence the payoffs of players of type 1 approach 0. If there is a small cost of entry c_1 for players 1, intuition suggests that in a steady state only a fraction of the new λ_1 players 1 will choose to enter,

[13]Rewritten in the form

$$v_i(\mu) = \frac{1}{1 - \delta_i} \sum_{j \in N} \pi_{ij}(\mu) \max\left(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu), 0\right), \forall i \in N,$$

the payoff formulae resemble the value equations (5)-(6) of Shimer and Smith (2000) (see also equation (14) in Smith (2011)) and Lemma 2 from Manea (2011).

and their probability of being matched for bargaining is sufficiently low so that they receive a payoff of c_1 in the game. Each of the new players 1 is then indifferent between entering the market and staying out.

Suppose that players of type i face a cost $c_i > 0$ to enter the market. When does a market distribution μ constitute a steady state of the economy with inflows $\lambda = (\lambda_i)_{i \in N}$ and entry costs $c = (c_i)_{i \in N}$? It must be that every player of type i enters the game if $v_i(\mu) > c_i$ and does not if $v_i(\mu) < c_i$. Moreover, the measure of new players joining population i in any period must be identical to the measure of players i who reach agreement that period. A concise definition of steady states can be expressed in terms of fixed points of the correspondence S, which we construct below. It is useful to first introduce the correspondence $X : \mathbb{R} \rightrightarrows [0,1]$,

$$X(a) = \begin{cases} \{0\} & \text{if } a < 0 \\ [0,1] & \text{if } a = 0 \\ \{1\} & \text{if } a > 0 \end{cases} \cdot$$

Definition 1. A market distribution μ constitutes a **steady state** for the economy with inflows λ and entry costs c if μ is a fixed point of the correspondence $S : [0,\infty)^n \setminus \{\mathbf{0}\} \rightrightarrows [0,\infty)^n \setminus \{\mathbf{0}\}$ defined by

$$(5.1) \quad S(\mu) = \left\{ (\mu_i - \sum_{j \in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) + \tilde{\lambda}_i)_{i \in N} \middle| \tilde{\beta}_{ij} \in \beta_{ij}(\mu) X\left(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu)\right) \ \& \ \tilde{\lambda}_i \in \lambda_i X\left(v_i(\mu) - c_i\right) \right\}.$$

In 5.1, the term $\mu_i - \sum_{j \in N}(\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) + \tilde{\lambda}_i$ represents the size of population i in the "next" period under the following premises

- μ_i is the measure of players i participating in the "current" market
- $\tilde{\beta}_{ij}$ is the mass of proposer-responder matches (i,j) that reach agreement in the current period
- $\tilde{\lambda}_i$ is the measure of new players i who enter the market in the next period.

The constraints on $\tilde{\beta}_{ij}$ and $\tilde{\lambda}_i$ reflect the equilibrium requirements on agreements and entry decisions *conditional on facing the stationary market* μ, which is strategically captured by the model with exogenous matching probabilities where $p_{ijt} = \pi_{ij}(\mu)$ for all $t \geq 0$. The rate of agreement for matched pairs of players (i, j) must be 0, 1, or any number in the interval $[0, 1]$ depending on whether the sum of their continuation payoffs $\delta_i v_i(\mu) + \delta_j v_j(\mu)$ is greater than, less than, or equal to s_{ij}, respectively. The proportion of players i entering the market is 0, 1, or some number in $[0, 1]$ if their payoff $v_i(\mu)$ is smaller than, larger than, or equal to c_i, respectively.

Remark 4. Note that $S(\mu)$ is not always a product set since $\tilde{\beta}_{ij}$ and $\tilde{\beta}_{ji}$ link its i and j components. While the projection of $S(\mu)$ on the i coordinate is

$$pr_i(S(\mu)) = \mu_i - \sum_{j \in N} (\beta_{ij}(\mu) + \beta_{ji}(\mu)) X\left(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu)\right) + \lambda_i X\left(v_i(\mu) - c_i\right),$$

we have $S(\mu) \neq \prod_{i \in N} pr_i(S(\mu))$ whenever there is a link ij with $\delta_i v_i(\mu) + \delta_j v_j(\mu) = s_{ij}$.

We show that if the matching technology satisfies a mild regularity assumption, steady states exist for any sufficiently low entry costs.

Theorem 3. *Suppose that the matching process satisfies the following condition*

$$\alpha := \inf_{\mu \in [0,\infty)^n \setminus \{\mathbf{0}\}} \max_{i,j \in N} \frac{\pi_{ij}(\mu)}{1 + \pi_{ij}(\mu)} s_{ij} > 0. \tag{5.2}$$

Then for any inflows $\lambda \in (0, \infty)^n$ and entry costs $c \in (0, \alpha]^n$, the economy has a steady state $\mu \in [0, \infty)^n \setminus \{\mathbf{0}\}$.

Condition 5.2 requires that for each market state there exists a player who, with probability uniformly bounded away from zero, gets the opportunity to share positive gains from trade with another player.[14] If for every $i \in N$ there exists $j \in N$ such that $s_{ij} > 0$, then the natural matching technology described by 2.3 satisfies 5.2, with

$$\alpha \geq \frac{p \min\{s_{ij} | i, j \in N; s_{ij} > 0\}}{2n + p}.$$

The proof of Theorem 3 shows that $S(\mu) \subset \mathcal{C}, \forall \mu \in \mathcal{C}$, where

$$\mathcal{C} = \left\{ \mu \in [0, \infty)^n \setminus \{\mathbf{0}\} \middle| \sum_{i \in N} \mu_i \geq \min_{i \in N} \lambda_i \ \&\ \mu_i \leq \lambda_i \left(1 + \frac{\max_{j \in N} s_{ij}}{c_i(1 - \delta_i)}\right), \forall i \in N \right\}.$$

Then it argues that the restriction of S to the set $\mathcal{C}$ satisfies the hypotheses of Kakutani's theorem and concludes that S has a fixed point in $\mathcal{C}$.

[14]The former player may belong to a population of size zero.

The definition of steady states takes the view that the initial players participate in the market for free or have a sunk cost of entry. Then some players present in the market "before" the first period may receive payoffs smaller than the entry cost for their types. One may wonder about the existence of steady states μ with the property that $\mu_i = 0$ if $v_i(\mu) < c_i$.[15] The next result provides a positive answer.

Theorem 3'. *Suppose that the matching process satisfies the condition 5.2. Then for any inflows* $\lambda \in (0,\infty)^n$ *and entry costs* $c \in (0,\alpha]^n$*, the economy has a steady state* $\mu \in [0,\infty)^n \setminus \{\mathbf{0}\}$ *with* $\mu_i = 0$ *whenever* $v_i(\mu) < c_i$*.*

The proof proceeds as follows. For $\rho \in [0,1]$, perturb S to obtain the correspondence $S^\rho : [0,\infty)^n \setminus \{\mathbf{0}\} \rightrightarrows [0,\infty)^n \setminus \{\mathbf{0}\}$,

$$(5.3)\quad S^\rho(\mu) = \left\{ \left(\rho \left(\mu_i - \sum_{j\in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) \right) + \tilde{\lambda}_i \right)_{i\in N} \middle| \tilde{\beta}_{ij} \in \beta_{ij}(\mu) X\left(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu)\right) \;\&\; \tilde{\lambda}_i \in \lambda_i X\left(v_i(\mu) - c_i\right) \right\}.$$

Fixed points of S^ρ describe steady states in an environment where the size of each population decays by a factor of ρ in every round (but players do not take into account their personal probability of exit). With minimal modifications, the argument from the proof of Theorem 3 extends to prove that S^ρ has a fixed point $\mu^\rho \in \mathcal{C}$ for each $\rho \in [0,1]$. It can be easily checked that $v_i(\mu^\rho) < c_i \Rightarrow \mu_i^\rho = 0$. In the Appendix, we show that $(\mu^\rho)_{\rho\in[0,1]}$ has a limit point as $\rho \to 1$, which constitutes a steady state with the desired property.

Theorems 3 and 3' do not provide intuition for the formation of steady states in which all populations have non-zero measure. We say that a steady state μ is **positive** if it assigns a positive size to each population and provides all players (weak) incentives to enter the market, i.e., $\mu_i > 0$ and $v_i(\mu) \geq c_i$ for all $i \in N$. The next example shows that some economies admit positive steady states only for specific configurations of entry costs.

Example 2. Consider a three-population setting with the matching technology given by 2.3. Assume that $s_{11} = s_{22} = s_{33} = s_{23} = 0 < s_{12} = s_{13}$ and $\lambda_1 < \lambda_2 = \lambda_3$. Let μ be a corresponding steady state market with $\mu_i > 0$ and $v_i(\mu) \geq c_i$ for all i. It is obvious that

[15]Note that if μ is a steady state with $\mu_i > 0$ and $v_i(\mu) < c_i$, then $v_i(\mu) = 0$.

agreements arise in the steady state equilibrium only when a player of type 1 is matched to some player of type 2 or 3. Since the measure λ_i of potential entrants of type $i = 2, 3$ exceeds the measure λ_1 of type 1 entrants, it must be that only a fraction of the λ_i players i enters, so $v_i(\mu) \leq c_i$. It follows that $v_i(\mu) = c_i$ for $i = 2, 3$. Note that the matching technology 2.3 assigns the same probability to every player of type 2 or 3 having the opportunity to make an offer to some player 1. Then $v_2(\mu) = v_3(\mu)$. Hence a positive steady state exists only if $c_2 = c_3$.[16]

In general, the costs of entry that guarantee the existence of positive steady states depend on the pattern of gains from trade and the matching process in a complex fashion. However, if we regard entry costs as an explanation for the operation of stationary markets, we may ask whether for given inflows λ there exist small costs c such that the resulting economy admits a positive steady state. The result below answers the latter question affirmatively, even for matching processes that violate the regularity condition 5.2.

Theorem 3". *For every vector of inflows $\lambda \in (0, \infty)^n$ and any $k > 0$, there exists $c \in [0, k]^n$ such that the economy with entry costs c has a positive steady state.*

Loosely speaking, the result shows that, for any continuous matching process, one can set arbitrarily low entry fees to limit the inflows into the populations that need to be rationed in order to attain a positive steady state.

Remark 5. Shimer and Smith (2000) prove the existence of steady states in a continuous time search model with a continuum of types. They impose some additional structure on the pattern of gains from trade and assume anonymous random matching. While our fixed point constructions have similar flavors, the two models are vastly different, and the existence proofs do not share many formal connections.

6. Conclusion

We analyzed a general model of bargaining in decentralized dynamic markets. The model features multiple populations that share heterogenous trading opportunities among them.

[16]The conclusion is not a consequence of the "non-generic" surplus distribution or matching technology. Indeed, a similar argument establishes that if $s_{21} > s_{31}$ and $\pi_{21}(\mu) > \pi_{31}(\mu), \forall \mu \in (0, \infty)^3$ in the setting of this example, then the existence of a positive steady state entails that $c_2 > c_3$.

The inflows of new players into each population are exogenous and possibly non-stationary. The state of the market at any date is determined by the size of the inflows and the volume of trade prior to that date. At every point in time, the matching probabilities for any pair of player types are endogenously derived from the underlying market distribution. In this setting, the bargaining powers of market participants coevolve over time in relation to the structure of agreements, the path of matching frequencies, and the overall trajectory of the economy. Our comprehensive framework provides insights into richer market dynamics than earlier models of bargaining in markets.

We established that an equilibrium always exists. We also proved that all equilibria leading to the same evolution of the economy are payoff equivalent. The unique equilibrium payoffs consistent with a given market path can be computed using an iterative method. However, the equilibrium payoffs are not necessarily unique. We showed by example that multiple self-fulfilling beliefs about the trajectory of the economy may coexist, giving rise to entirely different equilibrium dynamics.

A significant part of the previous literature on bargaining in markets focused on the (relatively more tractable) analysis of steady states. The current model provides a natural framework for investigating when and how steady states emerge. We offered theoretical foundations for the existence of steady states.

Appendix A. Proofs

Proof of Theorem 1. Every information set begins a bargaining game that can be described by some path of inflows $(\tilde{\lambda}_{it})_{i \in N, t \geq 0}$ with $\tilde{\lambda}_{i0} > 0$ for all $i \in N$. In order to specify behavior at each stage characterized by any such inflows $(\tilde{\lambda}_{it})_{i \in N, t \geq 0}$, we define the sets of paths of possible fractions of agreeing pairs, market distributions, matching probabilities, and feasible payoffs, respectively, as follows

$$
\begin{aligned}
\mathcal{A} &= \{(a_{ijt})_{i,j \in N, t \geq 0} | a_{ijt} \in [0,1], \forall i, j \in N, t \geq 0\} \\
\mathcal{M}^{\tilde{\lambda}} &= \{(\mu_{it})_{i \in N, t \geq 0} | \mu_0 = \tilde{\lambda}_0; \mu_{it} \in [0, \sum_{\tau=0}^{t} \tilde{\lambda}_{i\tau}], \forall i \in N, t \geq 1\} \\
\mathcal{P} &= \{(p_{ijt})_{i,j \in N, t \geq 0} | p_{ijt} \in [0,1], \forall i, j \in N, t \geq 0\} \\
\mathcal{V} &= \{(v_{it})_{i \in N, t \geq 0} | v_{it} \in [0, \max_{j \in N} s_{ij}], \forall i \in N, t \geq 0\}.
\end{aligned}
$$

Each of the four sets can be regarded as a topological vector space via a natural embedding in the space $\mathbb{R}^{\mathbb{N}}$ (the countable product of the set of real numbers) endowed with the standard product topology. Note that the product topology on $\mathbb{R}^{\mathbb{N}}$ is metrizable, so the characterizations of closed sets and continuous functions in terms of convergent sequences apply for each of the four sets (Theorem 2.40, [1]). The spaces $\mathcal{A}, \mathcal{M}^{\tilde{\lambda}}, \mathcal{P}, \mathcal{V}$ are compact by Tychonoff's theorem.

We construct the correspondence $f^{\tilde{\lambda}} : \mathcal{A} \rightrightarrows \mathcal{A}$ by composing the correspondence α and the functions $v^*, \pi, \kappa^{\tilde{\lambda}}$, where

$$\mathcal{A} \xrightarrow{\kappa^{\tilde{\lambda}}} \mathcal{M}^{\tilde{\lambda}} \xrightarrow{\pi} \mathcal{P} \xrightarrow{v^*} \mathcal{V} \overset{\alpha}{\rightrightarrows} \mathcal{A}.$$

Thus $f^{\tilde{\lambda}} = \alpha \circ v^* \circ \pi \circ \kappa^{\tilde{\lambda}}$, where π is given by 2.2[17] and v^* is derived from Theorem 2, while $\kappa^{\tilde{\lambda}}$ and α are defined below. We will show how the fixed points of $f^{\tilde{\lambda}}$ can be used to describe the equilibrium path at stages of the bargaining game where the market is in state $\tilde{\lambda}_0$ and the future inflows are given by $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots$

For any $a \in \mathcal{A}$, the sequence $\kappa^{\tilde{\lambda}}(a)$ describes the path of the market under the assumption that a fraction a_{ijt} of the pairs (i,j) matched to bargain at time t (with i playing the role of the proposer if $i \neq j$) reaches agreement. Hence $\kappa^{\tilde{\lambda}}(a)$ is recursively defined by

$$\begin{aligned} \kappa^{\tilde{\lambda}}_{i0}(a) &= \tilde{\lambda}_{i0}, \forall i \in N \\ \kappa^{\tilde{\lambda}}_{i(t+1)}(a) &= \kappa^{\tilde{\lambda}}_{it}(a) + \tilde{\lambda}_{i(t+1)} - \sum_{j \in N} \left(a_{ijt}\beta_{ijt}(\kappa^{\tilde{\lambda}}_t(a)) + a_{jit}\beta_{jit}(\kappa^{\tilde{\lambda}}_t(a)) \right), \forall i \in N, t \geq 0. \end{aligned}$$

For any $v \in \mathcal{V}$, the set $\alpha_{ijt}(v)$ consists of the possible rates of agreement among the proposer-responder pairs (i,j) matched at time t, assuming that bargaining proceeds as if the expected period $t+1$ payoffs (in case of disagreement) were given by v_{t+1}. In this scenario, the fraction of pairs (i,j) that reach agreement is $0, 1$, or any number in $[0,1]$ depending on whether $\delta_i v_{i(t+1)} + \delta_j v_{j(t+1)}$ is strictly greater, strictly smaller, or equal to s_{ij}, respectively. Thus

$$\alpha_{ijt}(v) = \begin{cases} \{0\} & \text{if } \delta_i v_{i(t+1)} + \delta_j v_{j(t+1)} > s_{ij} \\ [0,1] & \text{if } \delta_i v_{i(t+1)} + \delta_j v_{j(t+1)} = s_{ij} \\ \{1\} & \text{if } \delta_i v_{i(t+1)} + \delta_j v_{j(t+1)} < s_{ij} \end{cases}.$$

[17]Although $\pi(\mu)$ is not defined if $\mu_{it} = 0$ for some i and t, this will not become an issue because $\kappa^{\tilde{\lambda}}(\mathcal{A})$ does not contain such μ's.

Our first goal is to apply the Kakutani-Fan-Glicksberg theorem (Corollary 17.55, [1]) to establish that $f^{\tilde{\lambda}} = \alpha \circ v^* \circ \pi \circ \kappa^{\tilde{\lambda}}$ has a fixed point. We then show how fixed points of $f^{\tilde{\lambda}}$ translate into equilibrium behavior. Note that the definitions of $\kappa^{\tilde{\lambda}}$ and π, along with the continuity of β (assumed) and v^* (Theorem 2), imply that the function $v^* \circ \pi \circ \kappa^{\tilde{\lambda}}$ is continuous. Since the correspondence α has a closed graph, it follows that $f^{\tilde{\lambda}} = \alpha \circ (v^* \circ \pi \circ \kappa^{\tilde{\lambda}})$ also has a closed graph. Furthermore, $f^{\tilde{\lambda}}$ takes non-empty convex values because α does. Clearly, $\mathcal{A}$ is a non-empty compact convex subset of a topological vector space that is linearly homeomorphic to $\mathbb{R}^{\mathbb{N}}$; the latter is a locally convex Hausdorff space (Theorem 16.2, [1]). Thus $f^{\tilde{\lambda}} : \mathcal{A} \rightrightarrows \mathcal{A}$ satisfies all the hypotheses of the Kakutani-Fan-Glicksberg theorem, and it must have a fixed point.

We next demonstrate how fixed points of $f^{\tilde{\lambda}}$ map into equilibria of the bargaining game. For every $(\tilde{\lambda}_{it})_{i \in N, t \geq 0}$ with $\tilde{\lambda}_{i0} > 0$ for all $i \in N$, let $a^{\tilde{\lambda}}$ be a fixed point of $f^{\tilde{\lambda}}$. We will show that the strategy profile constructed below constitutes an equilibrium in which the market follows the path $\kappa^{\lambda}(a^{\lambda})$ and the payoffs are given by $v^*(\pi(\kappa^{\lambda}(a^{\lambda})))$. Under the constructed strategies, play can fall into several regimes, indexed by $\tilde{\lambda}$. We first define the strategies for each regime $\tilde{\lambda}$ and then specify the transition rule between regimes.

In the regime $\tilde{\lambda}$ play is as follows. When a player i is selected to make an offer to some player j at stage $\tau = 0, 1, \ldots$ of the regime, he offers $x := \delta_j v^*_{j(\tau+1)}(\pi(\kappa^{\tilde{\lambda}}(a^{\tilde{\lambda}})))$ if $a^{\tilde{\lambda}}_{ij\tau} > 0$ and declines to bargain otherwise. In stage τ of the regime, any player j accepts all offers strictly greater than x and rejects all offers strictly smaller than x. Furthermore, a proportion $a^{\tilde{\lambda}}_{ij\tau}$ of the players j receiving the (exact) offer x from some i accepts it.[18] Clearly, in regime $\tilde{\lambda}$, if players conform to the prescribed behavior, then the market follows the path $\kappa^{\tilde{\lambda}}(a^{\tilde{\lambda}})$.

The game commences in the regime λ and progresses as follows. While being in a regime $\tilde{\lambda}$, play stays in the regime as long as the market evolves along the path $\kappa^{\tilde{\lambda}}(a^{\tilde{\lambda}})$. After any stage of the regime $\tilde{\lambda}$ where the market distribution diverges from the latter path, play switches to a new regime $\tilde{\tilde{\lambda}}$. If divergence occurs at the beginning of period t in the game (as a consequence of a positive measure of players deviating from regime $\tilde{\lambda}$ in period $t-1$), and the game proceeds to a market μ_t (including the period t inflows λ_t), then the new regime is given by $\tilde{\tilde{\lambda}}_0 = \mu_t, \tilde{\tilde{\lambda}}_1 = \lambda_{t+1}, \tilde{\tilde{\lambda}}_2 = \lambda_{t+2}, \ldots$

[18]For the sake of the argument, we assume that if $a^{\tilde{\lambda}}_{ij\tau} = 1$ then all (as opposed to "almost all") players j accept the offer x from any i at stage τ.

The description of the strategies in regime $\tilde{\lambda}$ is incomplete, in that, for $a^{\tilde{\lambda}}_{ij\tau} \in (0,1)$, it does not specify which fraction $a^{\tilde{\lambda}}_{ij\tau}$ of players j must accept the stipulated offer from the players i at stage τ. One may be concerned that any concrete procedure selecting a set of agreements leads to heterogeneity in the expected payoffs of each population of players at stage τ, but it turns out that payoffs are not affected by the selection procedure.[19] More specifically, we establish that the expected payoffs in regime $\tilde{\lambda}$ are given by $v^*(\pi(\kappa^{\tilde{\lambda}}(a^{\tilde{\lambda}})))$, regardless of the unspecified details of the strategy profile. Since the focus of this part of the proof is on the regime $\tilde{\lambda}$, in an extreme abuse of notation we write v^* for $v^*(\pi(\kappa^{\tilde{\lambda}}(a^{\tilde{\lambda}})))$ and π for $\pi(\kappa^{\tilde{\lambda}}(a^{\tilde{\lambda}}))$. Let $\mathcal{U}_{i\tau}$ denote the range of expected payoffs for the players of type i participating at stage τ of regime $\tilde{\lambda}$ (possible under the collection of strategy profiles complying with the regime).

Each value in $\mathcal{U}_{i\tau}$ is obtained as an expectation over several types of payoffs, depending on the *outcome* for the particular *player* i in *stage* τ of the regime $\tilde{\lambda}$ as follows

- elements of $\delta_i \mathcal{U}_{i(\tau+1)}$, for situations in which the *player* does not reach an agreement (including events where he is not matched for bargaining at stage τ)
- $\delta_i v^*_{i(\tau+1)}$, in instances where the *player* accepts an offer
- $s_{ij} - \delta_j v^*_{j(\tau+1)}$, for cases in which the *player's* offer to j is accepted.

The terms $s_{ij} - \delta_j v^*_{j(\tau+1)}$ appear in the expectation with positive probability only if $a^{\tilde{\lambda}}_{ij\tau} > 0$. Since $a^{\tilde{\lambda}} \in f^{\tilde{\lambda}}(a^{\tilde{\lambda}}) = \alpha(v^*)$ by definition, the condition $a^{\tilde{\lambda}}_{ij\tau} > 0$ implies that $s_{ij} - \delta_j v^*_{j(\tau+1)} \geq \delta_i v^*_{i(\tau+1)}$. If the latter weak inequality holds with equality, then $s_{ij} - \delta_j v^*_{j(\tau+1)}$ simply enters the expectation as $\delta_i v^*_{i(\tau+1)}$. Otherwise, we have $s_{ij} - \delta_j v^*_{j(\tau+1)} > \delta_i v^*_{i(\tau+1)}$, so $a^{\tilde{\lambda}}_{ij\tau} = 1$, which implies that all players j accept the offer $\delta_j v^*_{j(\tau+1)}$ from any i at stage τ (see footnote 18). In this case, the value $s_{ij} - \delta_j v^*_{j(\tau+1)}$ is weighted in the expectation by the probability $\pi_{ij\tau}$ with which every i is selected to make an offer to some j in stage τ of the regime. To sum up, any payoff in $\mathcal{U}_{i\tau}$ can be represented as a convex combination of elements of $\delta_i \mathcal{U}_{i(\tau+1)}$, $\delta_i v^*_{i(\tau+1)}$, and terms $s_{ij} - \delta_j v^*_{j(\tau+1)}$, where the latter receive positive weight—equal to $\pi_{ij\tau}$—only if $s_{ij} - \delta_j v^*_{j(\tau+1)} > \delta_i v^*_{i(\tau+1)}$. Formally, for all $u \in \mathcal{U}_{i\tau}$, there exist $w \in \text{co}(\mathcal{U}_{i(\tau+1)})$ and $q \in [0,1]$

[19] Note that the "symmetric" treatment whereupon each player j accepts the offer with probability $a^{\tilde{\lambda}}_{ij\tau}$ is not feasible due to the (unavoidable) restriction to pure strategies.

such that

$$u = \sum_{\{j \in N | s_{ij} - \delta_j v^*_{j(\tau+1)} > \delta_i v^*_{i(\tau+1)}\}} \pi_{ij\tau} \left(s_{ij} - \delta_j v^*_{j(\tau+1)}\right) + \left(1 - q - \sum_{\{j \in N | s_{ij} - \delta_j v^*_{j(\tau+1)} > \delta_i v^*_{i(\tau+1)}\}} \pi_{ij\tau}\right) \delta_i v^*_{i(\tau+1)} + q\delta_i w.$$

Theorem 2 shows that

$$v^*_{i\tau} = \sum_{j \in N} \pi_{ij\tau} \max\left(s_{ij} - \delta_j v^*_{j(\tau+1)}, \delta_i v^*_{i(\tau+1)}\right) + \left(1 - \sum_{j \in N} \pi_{ij\tau}\right) \delta_i v^*_{i(\tau+1)},$$

which can be rewritten as

$$v^*_{i\tau} = \sum_{\{j \in N | s_{ij} - \delta_j v^*_{j(\tau+1)} > \delta_i v^*_{i(\tau+1)}\}} \pi_{ij\tau} \left(s_{ij} - \delta_j v^*_{j(\tau+1)}\right) + \left(1 - \sum_{\{j \in N | s_{ij} - \delta_j v^*_{j(\tau+1)} > \delta_i v^*_{i(\tau+1)}\}} \pi_{ij\tau}\right) \delta_i v^*_{i(\tau+1)}.$$

We immediately obtain that

$$\sup_{u \in \mathcal{U}_{i\tau}} |u - v^*_{i\tau}| \leq \sup_{w \in \mathrm{co}(\mathcal{U}_{i(\tau+1)}), q \in [0,1]} q\delta_i |w - v^*_{i(\tau+1)}| \leq \delta_i \sup_{u \in \mathcal{U}_{i(\tau+1)}} |u - v^*_{i(\tau+1)}|.$$

Iterating the inequalities above, and observing that the sequence $(v^*_{i\tau})_{s \geq 0}$ and the sets $(\mathcal{U}_{i\tau})_{s \geq 0}$ are uniformly bounded, we conclude that $\sup_{u \in \mathcal{U}_{i\tau}} |u - v^*_{i\tau}| = 0$, which means that $\mathcal{U}_{i\tau} = \{v^*_{i\tau}\}$, for all τ. Therefore, the constructed strategies yield expected payoffs of $v^*_{i\tau}$ to all players i present at stage τ of the regime $\tilde{\lambda}$.

We can finally prove that the constructed strategies constitute an equilibrium of the bargaining game. In any period of the game, a deviation from the associated regime by a single player (or a measure-zero set of players) does not trigger a regime change. Hence we only need to check incentives within each regime. Moreover, the single deviation principle applies to our setting. In light of the finding that the regime $\tilde{\lambda}$ yields payoffs $v^*(\pi(\kappa^{\tilde{\lambda}}(a^{\tilde{\lambda}})))$, we can easily check that no player has a profitable one-shot deviation from the strategies prescribed by regime $\tilde{\lambda}$. □

Proof of Theorem 2. (i) Define the sequences $(m^k_{it})_{i\in N,t\geq 0}$ and $(M^k_{it})_{i\in N,t\geq 0}$ recursively for $k = 0, 1, \ldots$ as follows

$$m^0_{it} = 0, M^0_{it} = \max_{j\in N} s_{ij} \tag{A.1}$$

$$m^{k+1}_{it} = \sum_{j\in N} p_{ijt} \max\left(s_{ij} - \delta_j M^k_{j(t+1)}, \delta_i m^k_{i(t+1)}\right) + \left(1 - \sum_{j\in N} p_{ijt}\right) \delta_i m^k_{i(t+1)} \tag{A.2}$$

$$M^{k+1}_{it} = \sum_{j\in N} p_{ijt} \max\left(s_{ij} - \delta_j m^k_{j(t+1)}, \delta_i M^k_{i(t+1)}\right) + \left(1 - \sum_{j\in N} p_{ijt}\right) \delta_i M^k_{i(t+1)}. \tag{A.3}$$

We refer to strategies that assign positive probability only to actions that survive iterated conditional dominance as "surviving strategies." We simultaneously establish the following claims by induction on k. Under all surviving strategies, in period t every player of type i

(1) rejects any offer smaller than $\delta_i m^k_{i(t+1)}$ (regardless of the identity of the proposer)
(2) has an expected payoff (at the beginning of the period) of at most M^k_{it}
(3) accepts any offer greater than $\delta_i M^k_{i(t+1)}$ (regardless of the identity of the proposer)
(4) does not make an offer greater than $\delta_j M^k_{j(t+1)}$ when matched to bargain with some player j.

For the base case $k = 0$, claims (1) and (2) hold trivially. We also note at this stage that claims (3) and (4) follow from (2) for all k. Suppose that claim (2) holds. Fix a period t information set where i receives some offer $x > \delta_i M^k_{i(t+1)}$. Any strategy whereupon i rejects the offer x in period t leads to a period $t+1$ expected payoff of at most $M^k_{i(t+1)}$ under the surviving strategies. Hence such strategies are conditionally dominated by accepting x at the information set under consideration. We now show that claim (3) implies (4). Let $y > \delta_j M^k_{j(t+1)}$, and consider all strategies under which i offers y to some j in period t at a particular information set. If, as per claim (3), j accepts every offer greater than $\delta_j M^k_{j(t+1)}$, then each of the latter strategies is conditionally dominated by any strategy that prescribes an offer in the interval $(\delta_j M^k_{j(t+1)}, y)$ at the given information set.

Therefore, we only need to prove the induction hypotheses (1) and (2) for step $k+1$, assuming that the four claims hold for all earlier steps. Consider a period t information set where some player i has to respond to an offer $x < \delta_i m^{k+1}_{i(t+1)}$. We argue that accepting the offer x is conditionally dominated for player i by the following plan of action for sufficiently small $\varepsilon > 0$. Player i rejects any offer received at every time $t' \geq t$. When selected to make an offer to some j at date $t' = t+1, t+2, \ldots, t+k+1$, player i offers $\delta_j M_{j(t'+1)}^{k+t+1-t'} + \varepsilon$ if $s_{ij} - \delta_j M_{j(t'+1)}^{k+t+1-t'} \geq \delta_i m_{i(t'+1)}^{k+t+1-t'}$ and declines to bargain otherwise. Player i declines to bargain when selected as a proposer after date $t+k+1$. Note that, by the induction hypothesis, all players j accept the offers $\delta_j M_{j(t'+1)}^{k+t+1-t'} + \varepsilon$ at time $t' = t+1, t+2, \ldots, t+k+1$. Then we can easily show by induction on k that the constructed strategy generates a period t payoff for i of $\delta_i m_{i(t+1)}^{k+1}$ as $\varepsilon \to 0$ under the surviving strategies for the opponents, so it conditionally dominates accepting x in period t for sufficiently small $\varepsilon > 0$.

We now show that all the surviving strategies deliver expected payoffs of at most M_{it}^{k+1} at the beginning of period t to the players of type i present in the game at that time. Consider a period t information set where i is given the opportunity to make an offer to j. By the induction hypothesis, player j rejects any offer lower than $\delta_j m_{j(t+1)}^k$. Moreover, when i declines to bargain or makes an offer that is rejected, he can expect a period $t+1$ payoff of at most $M_{i(t+1)}^k$ under the surviving strategies. Hence i cannot make an offer that generates an expected payoff greater than $\max\left(s_{ij} - \delta_j m_{j(t+1)}^k, \delta_i M_{i(t+1)}^k\right)$. By the induction hypothesis, any action of some player j specifying an offer above $\delta_i M_{i(t+1)}^k$ to i in period t is eliminated in the process of iterated conditional dominance. Also by the induction hypothesis, in all cases where i does not reach an agreement in period t, he enjoys a period $t+1$ expected payoff of at most $M_{i(t+1)}^k$. Therefore, i's date t payoff under the surviving strategies cannot exceed the expression from A.3, which defines M_{it}^{k+1}.

Our next goal is to show that the sequences $(m_{it}^k)_{k\geq 0}$ and $(M_{it}^k)_{k\geq 0}$ converge to a common limit. One can easily demonstrate by induction that for all $i \in N, t \geq 0$,

- the sequence $(m_{it}^k)_{k\geq 0}$ is increasing in k
- the sequence $(M_{it}^k)_{k\geq 0}$ is decreasing in k
- $\max_{j\in N} s_{ij} \geq M_{it}^k \geq m_{it}^k \geq 0$ for all $k \geq 0$.

Hence the sequences $(m_{it}^k)_{k\geq 0}$ and $(M_{it}^k)_{k\geq 0}$ are convergent. We now prove that they have the same limit.

Let $D^k = \sup_{i \in N, t \geq 0} M_{it}^k - m_{it}^k$. We have

$$\begin{aligned}
D^{k+1} &= \sup_{i \in N, t \geq 0} M_{it}^{k+1} - m_{it}^{k+1} \\
&= \sup_{i \in N, t \geq 0} \sum_{j \in N} p_{ijt} \max\left(s_{ij} - \delta_j m_{j(t+1)}^k, \delta_i M_{i(t+1)}^k\right) + \left(1 - \sum_{j \in N} p_{ijt}\right) \delta_i M_{i(t+1)}^k \\
&- \sum_{j \in N} p_{ijt} \max\left(s_{ij} - \delta_j M_{j(t+1)}^k, \delta_i m_{i(t+1)}^k\right) + \left(1 - \sum_{j \in N} p_{ijt}\right) \delta_i m_{i(t+1)}^k \\
&= \sup_{i \in N, t \geq 0} \sum_{j \in N} p_{ijt} \left[\max\left(s_{ij} - \delta_j m_{j(t+1)}^k, \delta_i M_{i(t+1)}^k\right) - \max\left(s_{ij} - \delta_j M_{j(t+1)}^k, \delta_i m_{i(t+1)}^k\right)\right] \\
&+ \left(1 - \sum_{j \in N} p_{ijt}\right) \delta_i \left[M_{i(t+1)}^k - m_{i(t+1)}^k\right] \\
&\leq \sup_{i \in N, t \geq 0} \sum_{j \in N} p_{ijt} \max(\delta_j (M_{j(t+1)}^k - m_{j(t+1)}^k), \delta_i (M_{i(t+1)}^k - m_{i(t+1)}^k) + \left(1 - \sum_{j \in N} p_{ijt}\right) \delta_i D^k \\
&\leq \max_{j \in N} \delta_j D^k,
\end{aligned}$$

where the first inequality uses the following fact.

Lemma 1. *For all* $w_1, w_2, w_3, w_4 \in \mathbb{R}$,

$$|\max(w_1, w_2) - \max(w_3, w_4)| \leq \max(|w_1 - w_3|, |w_2 - w_4|).$$

Proof of Lemma 1. Suppose $w_1 = \max(w_1, w_2, w_3, w_4)$. Then

$$|\max(w_1, w_2) - \max(w_3, w_4)| = w_1 - \max(w_3, w_4) \leq w_1 - w_3 \leq \max(|w_1 - w_3|, |w_2 - w_4|).$$

The proof is similar for the cases when w_2, w_3, or w_4 is equal to $\max(w_1, w_2, w_3, w_4)$. □

It follows that $D^k \leq (\max_{j \in N} \delta_j)^k D^0 = (\max_{j \in N} \delta_j)^k \max_{j, j' \in N} s_{jj'}$ for all $k \geq 0$. Therefore, for every $i \in N, t \geq 0$, we have

$$0 \leq M_{it}^k - m_{it}^k \leq (\max_{j \in N} \delta_j)^k \max_{j, j' \in N} s_{jj'}, \forall k \geq 0,$$

which implies that the sequences $(m_{it}^k)_{k \geq 0}$ and $(M_{it}^k)_{k \geq 0}$ have the same limit, denoted $v_{it}^*(p)$. We omit the parameter p in $v^*(p)$ until we address the issue of continuity with respect to p.

Recall that iterated conditional dominance predicts that in period t every player of type i rejects offers smaller than $\delta_i m^k_{i(t+1)}$ and accepts offers greater than $\delta_i M^k_{i(t+1)}$. Since

$$\lim_{k\to\infty} m^k_{i(t+1)} = \lim_{k\to\infty} M^k_{i(t+1)} = v^*_{i(t+1)},$$

it follows that only actions specifying that i reject offers smaller than $\delta_i v^*_{i(t+1)}$ and accept offers greater than $\delta_i v^*_{i(t+1)}$ at time t can survive iterated conditional dominance.

(iii) We first show that all belief-independent equilibria of the model with exogenous matching probabilities p yield payoffs $(v^*_{it}(p))_{i\in N, t\geq 0}$ and then establish equilibrium existence. Note that all actions used with positive probability in any belief-independent equilibrium must survive iterated conditional dominance. Then claim (2) in the proof by induction from part (i) demonstrates that each player i obtains an expected payoff of at most M^k_{it} at the beginning of period t in every equilibrium. In the inductive argument we also constructed a sequence of strategies for i that, under the surviving strategies of the opponents, generates a limit payoff for i of $m^{k+1}_{i(t+1)}$ at the beginning of period $t+1$. A simple reindexing of that construction leads to strategies that deliver a limit period t payoff of m^k_{it} to i. In every equilibrium, i must not find it profitable to deviate to any of the latter strategies, so his period t expected payoff should be at least m^k_{it}. Since $\lim_{k\to\infty} m^k_{it} = \lim_{k\to\infty} M^k_{it} = v^*_{it}$, the arguments above establish that in every belief-independent equilibrium, any player i participating in the game at the beginning of period t has an expected payoff of v^*_{it}.

(iv) Taking the limit $k \to \infty$ in A.2, we obtain the following system of equations for v^*

$$v^*_{it} = \sum_{j\in N} p_{ijt} \max\left(s_{ij} - \delta_j v^*_{j(t+1)}, \delta_i v^*_{i(t+1)}\right) + \left(1 - \sum_{j\in N} p_{ijt}\right) \delta_i v^*_{i(t+1)}. \tag{A.4}$$

Thus we showed indirectly that the system 3.1 has a bounded solution. Inequalities similar to those above demonstrate that any two payoff vectors v and v' that solve 3.1 must satisfy

$$\max_{i\in N} |v_{it} - v'_{it}| \leq \max_{j\in N} \delta_j \max_{i\in N} |v_{i(t+1)} - v'_{i(t+1)}|.$$

If v and v' are bounded, then we can easily conclude that $v = v'$. Therefore, v^* is the unique bounded solution to the system of equations 3.1.

(ii) We now prove the existence of belief-independent equilibria. We claim that the following strategy profile constitutes an equilibrium. When player i has the opportunity to make an offer to some player j in period t, he offers $\delta_j v^*_{j(t+1)}$ if $\delta_i v^*_{i(t+1)} + \delta_j v^*_{j(t+1)} \leq s_{ij}$

and declines to bargain otherwise. At time t, any player j accepts all offers greater than or equal to $\delta_j v^*_{j(t+1)}$ and rejects all strictly smaller offers. In what follows, we show that the strategies above generate expected payoffs of v^*_{it} for all players of type i present in period t of the game. Then one can easily see that the constructed strategies constitute an equilibrium (the single-deviation principle extends straightforwardly to the present setting).

Fix a player of type i participating in period t of the game after some history. Let q_{ijt} denote the probability that the latter player accepts an offer from players of type j under the strategies constructed above.[20] We rewrite the equations A.4 as follows

$$v^*_{it} = \sum_{\{j\in N|\delta_i v^*_{i(t+1)}+\delta_j v^*_{j(t+1)}\leq s_{ij}\}} \left(p_{ijt}(s_{ij}-\delta_j v^*_{j(t+1)}) + q_{ijt}\delta_i v^*_{i(t+1)}\right) + \left(1 - \sum_{\{j\in N|\delta_i v^*_{i(t+1)}+\delta_j v^*_{j(t+1)}\leq s_{ij}\}} (p_{ijt}+q_{ijt})\right)\delta_i v^*_{i(t+1)}.$$

Substituting the formula for $v^*_{i(t+1)}$ in the last term of the equation for v^*_{it}, then the formula for $v^*_{i(t+2)}$ in the last term of the proxy for $v^*_{i(t+1)}$, and so on, we find that v^*_{it} represents the expected period t-discounted value for a player of type i of a stochastic prize generated as follows. At each date $t' \geq t$, conditional on not having received a prize by that time, for every $j \in N$ with $\delta_i v^*_{i(t+1)} + \delta_j v^*_{j(t+1)} \leq s_{ij}$, the prizes $s_{ij} - \delta_j v^*_{j(t'+1)}$ and $\delta_i v^*_{i(t'+1)}$ are realized with respective probabilities $p_{ijt'}$ and $q_{ijt'}$ (all events are mutually exclusive; a prize is not awarded in period t' with conditional probability $1 - \sum_{\{j\in N|\delta_i v^*_{i(t'+1)}+\delta_j v^*_{j(t'+1)}\leq s_{ij}\}}(p_{ijt'}+q_{ijt'})$). Note that the strategies constructed above lead to the same distribution over outcomes for the fixed player i from the perspective of period t as the stochastic prize. Hence the constructed strategies yield expected payoffs of v^*_{it} for all players of type i present in period t, as claimed.

(v) To show that $v^*_{it}(p)$ varies continuously in p, fix $\varepsilon > 0$ and let k be such that

$$(\max_{j\in N} \delta_j)^k \max_{j,j'\in N} s_{jj'} < \varepsilon/3.$$

The definition of M^k_{it} relies on the matching probabilities p, and we instate the notation $M^k_{it}(p)$ to underline this dependence. The resulting function M^k_{it} is obviously continuous in

[20]As footnote 11 asserts, the model with exogenous matching probabilities does not impose any restrictions on the probability that each player is chosen as the responder to a potential offer from a certain population of players. Hence q_{ijt} is derived from the (unspecified) underlying matching procedure in the particular game form under consideration and the constructed strategies for a *given* player i. The argument applies independently to every player of type i.

p. Then any given p has a neighborhood P such that

$$|M_{it}^k(p) - M_{it}^k(p')| < \varepsilon/3, \forall p' \in P.$$

Earlier arguments show that for all $p' \in P$,

$$v_{it}^*(p') \in [m_{it}^k(p'), M_{it}^k(p')]$$
$$0 \leq M_{it}^k(p') - v_{it}^*(p') \leq M_{it}^k(p') - m_{it}^k(p') \leq (\max_{j \in N} \delta_j)^k \max_{j,j' \in N} s_{jj'} < \varepsilon/3.$$

It follows that

$$|v_{it}^*(p) - v_{it}^*(p')| \leq |v_{it}^*(p) - M_{it}^k(p)| + |M_{it}^k(p) - M_{it}^k(p')| + |M_{it}^k(p') - v_{it}^*(p')| < \varepsilon, \forall p' \in P,$$

which completes the proof of continuity. □

Proof of Proposition 1. It is useful to explore the properties of the two types of equilibria for a given δ and varying x, and then apply the findings in the context of a fixed x and changing δ.

Equilibrium analysis for fixed δ and variable x.

All-agreement equilibria. We first inquire into the existence of all-agreement equilibria for initial markets with index $x \in [1/2, 1)$. As argued in the text, the market index must be x in every period along the equilibrium path. Moreover, the proofs of Theorems 1 and 2 show that the equilibrium payoffs are unique and stationary. The payoffs $(u_1(x), u_2(x))$ for the two populations solve the linear system

$$\begin{aligned} u_1(x) &= \frac{x}{4}(a - \delta u_1(x)) + \frac{1-x}{4}(1 - \delta u_2(x)) + \frac{3}{4}\delta u_1(x) \\ u_2(x) &= \frac{x}{4}(1 - \delta u_1(x)) + \frac{1-x}{4}(1 - \delta u_2(x)) + \frac{3}{4}\delta u_2(x). \end{aligned}$$

The unique solution of the system is

$$\begin{aligned} u_1(x) &= \frac{1}{2(2-\delta)} - \frac{\delta x^2(a-1)}{2(2-\delta)(4-3\delta)} + \frac{x(a-1)}{4-3\delta} \\ u_2(x) &= \frac{1}{2(2-\delta)} - \frac{\delta x^2(a-1)}{2(2-\delta)(4-3\delta)}. \end{aligned}$$

In order to provide incentives for the desired agreements and disagreements, we need $u_1(x) \geq 0, u_2(x) \geq 0, 2\delta u_1(x) \leq a, \delta(u_1(x) + u_2(x)) \leq 1, 2\delta u_2(x) \leq 1$. One can show that the inequalities $u_2(x) \geq 0$ and $\delta(u_1(x) + u_2(x)) \leq 1$ imply all the others for every $x \in [1/2, 1)$.[21]

Note that $u_2(x)$ is decreasing in x, so

$$u_2(x) \geq \lim_{y \to 1} u_2(y) = \frac{4 - (2+a)\delta}{2(2-\delta)(4-3\delta)} > 0,$$

as by assumption, $\delta < 1, a \leq 2$. Thus an all-agreement equilibrium exists if $\delta(u_1(x)+u_2(x)) \leq 1$. To study the latter inequality, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1 - \delta(u_1(x) + u_2(x))$.

We have

$$f(1/2) = \frac{8 - (7+a)\delta}{4(2-\delta)} < 0,$$

given the assumption that $\delta > 8/(7+a)$. Also,

$$\lim_{y \to 1} f(y) = \frac{2(1-\delta)(4-(2+a)\delta)}{(2-\delta)(4-3\delta)} > 0,$$

since $(2+a)\delta < 4$ for $\delta < 1, a \leq 2$. Since f is a quadratic function with a positive leading coefficient, there exists $\underline{x} \in (1/2, 1)$ such that $f(\underline{x}) = 0$, $f(x) > 0$ for $x \in (\underline{x}, 1)$ and $f(x) < 0$ for $x \in [1/2, \underline{x})$. Therefore, an all-agreement equilibrium exists for all $x \in [\underline{x}, 1)$.

Population-agreement equilibria. We next look for population-agreement equilibria. If the period t market distribution is μ_t, with a corresponding index $x = \mu_{1t}/(\mu_{1t} + \mu_{2t})$, and agreements arise as postulated, then the next period market is given by

$$\mu_{i(t+1)} = \mu_{it} \left(1 - 2\frac{\mu_{it}}{4(\mu_{1t} + \mu_{2t})}\right) \ (i = 1, 2),$$

with an index

$$\frac{\mu_{1(t+1)}}{\mu_{1(t+1)} + \mu_{2(t+1)}} = \frac{x(2-x)}{1 + 2x(1-x)} =: \tau(x).$$

One can easily check that $\tau(x) \in [1/2, 1)$ and $\tau(x) \leq x$ for all $x \in [1/2, 1)$. The function $\tau : [1/2, 1) \to [1/2, 1)$ has the following properties:

[21]Since

$$u_1(x) - u_2(x) = \frac{x(a-1)}{4-3\delta} > 0,$$

the following conditions hold $u_2(x) \geq 0 \Rightarrow u_1(x) \geq 0$ and $\delta(u_1(x) + u_2(x)) \leq 1 \Rightarrow 2\delta u_2(x) \leq 1$. To see that $\delta(u_1(x) + u_2(x)) \leq 1$ implies $2\delta u_1(x) \leq a$, note that the former inequality leads to

$$2\delta u_1(x) \leq 1 + \delta(u_1(x) - u_2(x)) = 1 + \delta\frac{x(a-1)}{4-3\delta} < a.$$

Indeed, the last inequality is equivalent to $\delta(x+3) < 4$, which holds for all $\delta < 1, x < 1$.

- τ is strictly increasing and continuous on $[1/2, 1)$ and has an inverse $\tau^{-1} : [1/2, 1) \to [1/2, 1)$
- for every $x \in [1/2, 1)$, the sequence $(\tau^k(x))_{k\geq 0}$ is decreasing and converges to $1/2$, which is the unique fixed point of τ on $[1/2, 1)$
- for every $x \in (1/2, 1)$, the sequence $(\tau^{-k}(x))_{k\geq 0}$ is increasing and converges to 1.[22]

We will show that for $x \in [1/2, \tau^{-1}(\underline{x})]$ there exists a population-agreement equilibrium. Theorems 1 and 2 show that, in such an equilibrium, the expected payoffs for players of type 1 and 2 in a period where the market index is x are given by, respectively,

$$\begin{aligned} v_1(x) &= a\sum_{k\geq 0} \delta^k \left(1 - \frac{x}{2}\right)\left(1 - \frac{\tau(x)}{2}\right)\cdots\left(1 - \frac{\tau^{k-1}(x)}{2}\right)\frac{\tau^k(x)}{4} \\ v_2(x) &= \sum_{k\geq 0} \delta^k \frac{1+x}{2}\ \frac{1+\tau(x)}{2}\cdots\frac{1+\tau^{k-1}(x)}{2}\ \frac{1-\tau^k(x)}{4}. \end{aligned}$$

To prove this claim, note that the function v_1 satisfies the following equations

$$\begin{aligned} v_1(x) &= a\left(\frac{x}{4} + \sum_{k\geq 1} \delta^k \left(1 - \frac{x}{2}\right)\left(1 - \frac{\tau(x)}{2}\right)\cdots\left(1 - \frac{\tau^{k-1}(x)}{2}\right)\frac{\tau^k(x)}{4}\right) \\ &= \frac{x}{4}a + \left(1 - \frac{x}{2}\right)\delta a\sum_{k\geq 0} \delta^k \left(1 - \frac{\tau(x)}{2}\right)\cdots\left(1 - \frac{\tau^k(x)}{2}\right)\frac{\tau^{k+1}(x)}{4} \\ &= \frac{x}{4}a + \left(1 - \frac{x}{2}\right)\delta v_1(\tau(x)) \\ &= \frac{x}{4}\left(a - \delta v_1(\tau(x))\right) + \left(1 - \frac{x}{4}\right)\delta v_1(\tau(x)). \end{aligned}$$

Similarly,

$$v_2(x) = \frac{1-x}{4}(1 - \delta v_2(\tau(x))) + \frac{3+x}{4}\delta v_2(\tau(x)).$$

We can construct equilibria in which the two types of players have expected payoffs $(v_1(x), v_2(x))$ in periods with market index $x \in [1/2, \tau^{-1}(\underline{x})]$ if the conjectured structure of agreements and disagreements is incentive compatible, i.e.,

$$\forall x \in [1/2, \tau^{-1}(\underline{x})]:\ 2\delta v_1(\tau(x)) \leq a,\ 2\delta v_2(\tau(x)) \leq 1,\ \delta\left(v_1(\tau(x)) + v_2(\tau(x))\right) \geq 1.$$

Through a change of variable, the latter condition becomes

$$\forall x \in [1/2, \underline{x}]:\ 2\delta v_1(x) \leq a,\ 2\delta v_2(x) \leq 1,\ \delta\left(v_1(x) + v_2(x)\right) \geq 1. \tag{A.5}$$

[22] τ^k (τ^{-k}) denotes τ's (τ^{-1}'s) composition with itself k times (by convention, τ^0 is the identity function).

A range of x where a population-agreement equilibrium exists. The first inequality in A.5 is a consequence of

$$\begin{aligned} v_1(x) &= a\sum_{k\geq 0}\delta^k\left(1-\frac{x}{2}\right)\left(1-\frac{\tau(x)}{2}\right)\ldots\left(1-\frac{\tau^{k-1}(x)}{2}\right)\frac{\tau^k(x)}{4} \\ &\leq a/2\sum_{k\geq 0}\left(1-\frac{x}{2}\right)\left(1-\frac{\tau(x)}{2}\right)\ldots\left(1-\frac{\tau^{k-1}(x)}{2}\right)\frac{\tau^k(x)}{2} \\ &= a/2. \end{aligned}$$

The second inequality is proven analogously.

We are left to establish that $\delta(v_1(x)+v_2(x)) \geq 1$ holds for all $x \in [1/2, \underline{x}]$. Note that $\tau^k(1/2) = 1/2$ for all $k \geq 0$. Then $v_1(1/2) = a\sum_{k\geq 0}\delta^k(3/4)^k(1/8) = a/(8-6\delta)$, and analogously $v_2(1/2) = 1/(8-6\delta)$. Hence $\delta(v_1(1/2)+v_2(1/2)) = \delta(a+1)/(8-6\delta) > 1$ for $\delta > 8/(7+a)$. Clearly, $v_1(x)$ and $v_2(x)$ vary continuously in x, so there exists $x_0 \in (1/2, \underline{x})$ such that $\delta(v_1(x)+v_2(x)) > 1$ for all $x \in [1/2, x_0]$.

Define $x_k = \tau^{-k}(x_0)$ for $k \geq 1$. As stated earlier, the sequence $(x_k)_{k\geq 0}$ is increasing and converges to 1 as $k \to \infty$. We prove by induction on k that $\delta(v_1(x)+v_2(x)) > 1$ for all $x \in [1/2, \min(x_k, \underline{x})]$. Note that we have already established the claim for the base case $k = 0$. We now assume that the claim is true over the interval $[1/2, \min(x_{k-1}, \underline{x})]$ and show that it holds over $[1/2, \min(x_k, \underline{x})]$.

Fix $x \in [1/2, \min(x_k, \underline{x})]$. For the purposes of proving the induction step, we abuse notation and write v_i for $v_i(x)$, v_i' for $v_i(\tau(x))$, and u_i for $u_i(\underline{x})$ $(i = 1, 2)$. The goal is thus to show that $\delta(v_1+v_2) > 1$.

Since $x \in [1/2, \min(x_k, \underline{x})]$, we have that $\tau(x) \leq \tau(\min(x_k, \underline{x})) = \min(x_{k-1}, \tau(\underline{x})) \leq \min(x_{k-1}, \underline{x})$. Hence the induction hypothesis implies that $\delta(v_1'+v_2') > 1$.

The earlier payoff equations can be rewritten as follows

$$\begin{aligned} v_1 &= \frac{x}{4}a + \left(1-\frac{x}{2}\right)\delta v_1' \\ v_2 &= \frac{1-x}{4} + \frac{1+x}{2}\delta v_2' \\ u_1 &= \frac{\underline{x}}{4}a + \left(1-\frac{\underline{x}}{2}\right)\delta u_1 \\ u_2 &= \frac{1-\underline{x}}{4} + \frac{1+\underline{x}}{2}\delta u_2. \end{aligned}$$

The last two identities use the fact that $f(\underline{x}) = 1-\delta(u_1+u_2) = 0$.

We set out to show that $\delta(v_1+v_2) > \delta(u_1+u_2) = 1$, or equivalently that $v_1+v_2-u_1-u_2 > 0$. Manipulating the identities above, we obtain

$$\begin{aligned}
\text{(A.6)}\quad & v_1+v_2-u_1-u_2 \\
&= \frac{x-\underline{x}}{4}(a-1) + \left(1-\frac{x}{2}\right)\delta v_1' - \left(1-\frac{\underline{x}}{2}\right)\delta u_1 + \frac{1+x}{2}\delta v_2' - \frac{1+\underline{x}}{2}\delta u_2 \\
&= \frac{x-\underline{x}}{4}(a-1) + \left(1-\frac{x}{2}\right)\delta(v_1'-u_1) + \frac{\underline{x}-x}{2}\delta u_1 + \frac{1+x}{2}\delta(v_2'-u_2) + \frac{x-\underline{x}}{2}\delta u_2 \\
&= \frac{\underline{x}-x}{4}\left(2\delta(u_1-u_2)-(a-1)\right) + \left(1-\frac{x}{2}\right)\delta(v_1'+v_2'-u_1-u_2) + \left(x-\frac{1}{2}\right)\delta(v_2'-u_2).
\end{aligned}$$

We show that every term in the last sum is non-negative, with the second one being positive. Since $x \in [1/2, \min(x_k, \underline{x})]$ and $\underline{x} < 1$, the coefficients satisfy the following inequalities $\underline{x} - x \geq 0, 1 - x/2 > 0, x - 1/2 \geq 0$. The second term is positive since we argued that $\delta(v_1' + v_2') > 1 = \delta(u_1 + u_2)$.

To show that the first term is non-negative, we need to prove that $2\delta(u_1-u_2)-(a-1) \geq 0$, which can be rewritten as $u_1 - u_2 \geq (a-1)/(2\delta)$, or

$$\frac{\underline{x}(a-1)}{4-3\delta} \geq \frac{a-1}{2\delta}.$$

The latter inequality is equivalent to $\underline{x} \geq 2/\delta - 3/2$. Since $2/\delta - 3/2 > 1/2$, using the properties of the function f discussed earlier, $\underline{x} \geq 2/\delta - 3/2$ is equivalent to $f(2/\delta - 3/2) \leq 0$. We find that, if $\delta > 8/(7+a)$, then

$$f(2/\delta - 3/2) = \frac{8-\delta(7+a)}{4(2-\delta)} < 0.$$

The third term is non-negative because

$$\text{(A.7)}\quad v_2' = \sum_{k\geq 0} \delta^k \frac{1+\tau(x)}{2}\,\frac{1+\tau^2(x)}{2} \cdots \frac{1+\tau^k(x)}{2}\,\frac{1-\tau^{k+1}(x)}{4} \geq \sum_{k\geq 0} \delta^k \left(\frac{1+\underline{x}}{2}\right)^k \frac{1-\underline{x}}{4} = u_2.$$

For a proof, note that the first sum represents the expected value of a random variable generated as follows. A coin is tossed at every date $k = 0, 1, \ldots$ until a heads outcome is observed. The conditional probability of heads turning up at time k is $(1-\tau^{k+1}(x))/2$. In the event that the first heads appears at date k, the realized discounted payoff is $\delta^k/2$. Similarly, the second sum can be interpreted as the present value of an analogous process where heads is obtained with probability $(1-\underline{x})/2$ at each date. The inequality follows from the fact that the distribution of the former random variable first-order stochastically dominates that of the latter ($\tau^{k+1}(x) \leq x \leq \underline{x}$ for $x \in [1/2, \min(x_k, \underline{x})]$ and $k \geq 0$).

We proved the existence of the two types of equilibria for the bargaining game with an initial market indexed by $x \in [\underline{x}, \tau^{-1}(\underline{x})]$. The all-agreement equilibrium delivers expected payoffs $(u_1(x), u_2(x))$, while the population-agreement one leads to payoffs $(v_1(x), v_2(x))$.

Equilibrium analysis for fixed x and variable δ. We next explore the existence of the two types of equilibria for a given $x \in [1/2, 1)$, as we vary $\delta \in [0, 1)$, to prove each part of Proposition 1. We revise the notation to recognize that $u_i(x), v_i(x), f(x)$ depend on δ and write $u_i(x,\delta), v_i(x,\delta), f(x,\delta)$ instead ($i = 1, 2$).

Part (i). As already argued, an all-agreement equilibrium exists if and only if

$$f(x,\delta) = \frac{8 - 2\delta(7 + (a-1)x) + \delta^2(6 + (a-1)x(x+1))}{(2-\delta)(4-3\delta)} \geq 0.$$

The inequality above is equivalent to

$$g(x,\delta) := 8 - 2\delta(7 + (a-1)x) + \delta^2(6 + (a-1)x(x+1)) \geq 0.$$

Note that g is a quadratic function in the second variable with a positive leading coefficient and $g(x,1) = -(a-1)x(1-x) < 0$. It follows that for every $x \in [1/2, 1)$ there exists $\bar{\delta}(x)$ such that $g(x,\delta) \geq 0$ (for $\delta \in [0,1)$) if and only if $\delta \leq \bar{\delta}(x)$. Moreover,

$$g\left(x, \frac{8}{7+a}\right) = \frac{32(a-1)(2x-1)}{(7+a)^2}\left(x - \frac{a+1}{4}\right)$$

implies that $\bar{\delta}(x) > 8/(7+a)$ for $x > (a+1)/4$.

The population-agreement equilibrium exits if and only if

$$h(x,\delta) := 1 - \delta(v_1(x,\delta) + v_2(x,\delta)) \leq 0.$$

Since $v_1(x,\delta)$ and $v_2(x,\delta)$ are continuous and increasing in δ, the function h is continuous and decreasing in the second argument. Then $h(x,0) = 1 > 0 > h(x,1) = (1-a)/2$ implies the existence of $\underline{\delta}(x)$ such that $h(x,\delta) \leq 0$ if and only if $\delta \geq \underline{\delta}(x)$.

Part (ii). We next show that $\bar{\delta}(x) > \underline{\delta}(x)$ for all $x > (a+1)/4$. The arguments from the first stage of the proof, applied with x playing the role of $\underline{x}$ and $\delta = \bar{\delta}(x) > 8/(7+a)$, establish that

$$\bar{\delta}(x)\left(v_1(x,\bar{\delta}(x)) + v_2(x,\bar{\delta}(x))\right) > \bar{\delta}(x)\left(u_1(x,\bar{\delta}(x)) + u_2(x,\bar{\delta}(x))\right) = 1.$$

The definition of $\underline{\delta}(x)$, along with the continuity and strict monotonicity of $h(x,\cdot)$, leads to $\bar{\delta}(x) > \underline{\delta}(x)$.

Part (iii). Consider a pair (x,δ) for which both types of equilibria exist. As argued earlier, the unique payoffs (u_1,u_2) for the two populations in the all-agreement equilibrium satisfy the conditions

$$\begin{aligned} u_1 &= \frac{x}{4}(a-\delta u_1) + \frac{1-x}{4}(1-\delta u_2) + \frac{3}{4}\delta u_1 \\ u_2 &= \frac{x}{4}(1-\delta u_1) + \frac{1-x}{4}(1-\delta u_2) + \frac{3}{4}\delta u_2 \\ \delta(u_1+u_2) &\leq 1. \end{aligned}$$

Since $1-\delta u_2 \geq \delta u_1$, we have

$$u_1 \geq \frac{x}{4}(a-\delta u_1) + \left(1-\frac{x}{4}\right)\delta u_1 = \frac{x}{4}a + \left(1-\frac{x}{2}\right)\delta u_1,$$

which leads to

$$u_1 \geq a\sum_{k\geq 0}\delta^k\left(1-\frac{x}{2}\right)^k\frac{x}{4}.$$

On the other hand, the payoffs (v_1,v_2) in the population-agreement equilibrium satisfy

$$\begin{aligned} v_1 &= a\sum_{k\geq 0}\delta^k\left(1-\frac{x}{2}\right)\left(1-\frac{\tau(x)}{2}\right)\cdots\left(1-\frac{\tau^{k-1}(x)}{2}\right)\frac{\tau^k(x)}{4} \\ v_2 &= \sum_{k\geq 0}\delta^k\frac{1+x}{2}\,\frac{1+\tau(x)}{2}\cdots\frac{1+\tau^{k-1}(x)}{2}\,\frac{1-\tau^k(x)}{4} \\ 1 &\leq \delta(v_1+v_2). \end{aligned}$$

An argument similar to the one for A.7 establishes that $u_1 \geq v_1$. Then the inequalities $\delta(u_1+u_2) \leq 1 \leq \delta(v_1+v_2)$ imply that $u_2 \leq v_2$.

Part (iv). Let $U(x,\delta)$ and $V(x,\delta)$ denote the total welfare attained in the bargaining game with an initial measure x of players 1 and $1-x$ of players 2, sharing the discount factor δ, if agreements arise as in the all-agreement and population-agreement equilibria, respectively. U solves the following equation[23]

$$U(x,\delta) = a\frac{x^2}{4} + \frac{x(1-x)}{2} + \frac{(1-x)^2}{4} + \frac{1}{2}\delta U(x,\delta).$$

Thus

$$U(x,\delta) = \frac{(a-1)x^2+1}{2(2-\delta)}.$$

Similarly, V satisfies the formula

$$V(x,\delta) = a\frac{x^2}{4} + \frac{(1-x)^2}{4} + \left(\frac{1}{2} + x(1-x)\right)\delta V(\tau(x),\delta).$$

To obtain bounds on $V(x,\delta)$, note that if the expression

$$D(y,\delta) := V(y,\delta) - \left(\frac{1}{2} + y(1-y)\right)\delta V(\tau(y),\delta) - \left(U(y,\delta) - \left(\frac{1}{2} + y(1-y)\right)\delta U(\tau(y),\delta)\right)$$

is positive (negative) for all $y \in (1/2, x]$, then we can immediately conclude that $V(x,\delta)$ is strictly greater (smaller) than $U(x,\delta)$.

Using the formula for $U(x,\delta)$ and the recursion for $V(\cdot,\delta)$, we compute

$$D(y,\delta) = \frac{y(1-y)(4+(5+3a)y(1-y))}{4(2-\delta)(1+2y(1-y))}\left(\delta - \frac{4+8y(1-y)}{4+(5+3a)y(1-y)}\right)$$

Hence $D(y,\delta)$ is positive (negative) for all $y \in (1/2, x]$ if

$$\delta > \ (<) \ \frac{4+8y(1-y)}{4+(5+3a)y(1-y)} =: d(y), \forall y \in (1/2, x].$$

Since $d(y)$ is strictly increasing in y for $y \in (1/2, x]$, we have that

$$\begin{aligned} \delta > d(x) \quad &\Rightarrow \quad V(x,\delta) > U(x,\delta) \\ \delta \leq \lim_{y\to 1/2} d(y) = \frac{8}{7+a} \quad &\Rightarrow \quad V(x,\delta) < U(x,\delta). \end{aligned}$$

[23]In a market with x players of type 1 and $1-x$ players of type 2, there is a mass of $x^2/4$ pairs of players 1 matched to bargain with one another, $2\times x(1-x)/4$ pairs of players of types 1 and 2, and $(1-x)^2/4$ pairs of players 2. The measures of players of type 1 and 2 left unmatched in the first period are $x - \left(2\times x^2/4 + x(1-x)/2\right) = x/2$ and $1-x-\left(2\times(1-x)^2/4 + x(1-x)/2\right) = (1-x)/2$, respectively. If all first period matches result in agreement, the second period market contains half of the players in each population and contributes to welfare with a surplus of $\delta U(x,\delta)/2$.

We showed that if $\bar{\delta}(x) > d(x)$ then $V(x, \bar{\delta}(x)) > U(x, \bar{\delta}(x))$. The inequality $\bar{\delta}(x) > d(x)$ is equivalent to

$$g(x, d(x)) = \frac{24(a-1)x^2(2-x)(1-x)(2x-1)}{(4+(5+3a)x(1-x))^2}\left(\frac{(x+1)(7x-4x^2-1)}{3x(2-x)} - a\right) > 0.$$

Thus $V(x, \bar{\delta}(x)) > U(x, \bar{\delta}(x))$ whenever

$$\frac{(x+1)(7x-4x^2-1)}{3x(2-x)} > a.$$

For every $a \in (1, 4/3)$, there exists $\varepsilon > 0$ such that the inequality above holds for all $x \in (1-\varepsilon, 1)$, as

$$\lim_{x \to 1} \frac{(x+1)(7x-4x^2-1)}{3x(2-x)} = 4/3.$$

Consider now $\tilde{x} = (a+1)/4$. We have $\bar{\delta}(\tilde{x}) = 8/(7+a)$, and the discussion above proves that $V(\tilde{x}, \bar{\delta}(\tilde{x})) < U(\tilde{x}, \bar{\delta}(\tilde{x}))$. Since $U, V, \bar{\delta}$ are continuous functions on their respective domains, it follows that there exists $\varepsilon > 0$ such that $V(x, \bar{\delta}(x)) < U(x, \bar{\delta}(x))$ for all $x \in ((a+1)/4, (a+1)/4 + \varepsilon)$. □

Proof of Theorem 3. Note that $v(\mu) = v^*\left((p_{ijt})_{i,j \in N, t \geq 0}\right)$, where $p_{ijt} = \pi_{ij}(\mu)$ for all $i, j \in N, t \geq 0$. Since v^* is a continuous function by Theorem 2 and π is also continuous, it follows that the steady state payoffs $v(\mu)$ vary continuously for $\mu \in [0,\infty)^n \setminus \{\mathbf{0}\}$. The continuity of v and β, along with the closedness of X's graph, implies that S has a closed graph on $[0,\infty)^n \setminus \{\mathbf{0}\}$ (endowed with the relative standard topology).

Clearly, $S(\mu)$ is a non-empty convex set for every $\mu \in [0,\infty)^n \setminus \{\mathbf{0}\}$. Aiming to apply Kakutani's fixed point theorem, we seek to restrict the domain and range of S to a compact convex subset of an Euclidean space. Define

$$\begin{aligned} M_i &= \lambda_i\left(1 + \frac{\max_{j \in N} s_{ij}}{c_i(1-\delta_i)}\right) \\ \mathcal{C} &= \left\{\mu \in [0,\infty)^n \setminus \{\mathbf{0}\} \middle| \sum_{i \in N} \mu_i \geq \min_{i \in N} \lambda_i; \mu_i \leq M_i, \forall i \in N\right\}. \end{aligned}$$

$\mathcal{C}$ is a non-empty, compact, and convex subset of $\mathbb{R}^n$. In what follows, we establish that $S(\mu) \subset \mathcal{C}, \forall \mu \in \mathcal{C}$. This step constitutes the core of our fixed point argument.

We first show that for every market state $\mu \in [0,\infty)^n \setminus \{\mathbf{0}\}$ we have $\sum_{i \in N} \tilde{\mu}_i \geq \min_{i \in N} \lambda_i, \forall \tilde{\mu} \in S(\mu)$. Since $v_i(\mu) > c_i \Rightarrow \tilde{\mu}_i \geq \lambda_i, \forall \tilde{\mu} \in S(\mu)$, it suffices to argue that for every μ there exists

$i \in N$ such that $v_i(\mu) > c_i$. We prove the latter assertion by contradiction. Suppose that

$$v_i(\mu) \leq c_i, \forall i \in N$$

for some market state μ. Let

$$(\bar{i}, \bar{j}) \in \arg\max_{(i,j) \in N \times N} \frac{\pi_{ij}(\mu)}{1 + \pi_{ij}(\mu)} s_{ij}.$$

By assumption,

$$0 < c_i \leq \alpha \leq \frac{\pi_{\bar{i}\bar{j}}(\mu)}{1 + \pi_{\bar{i}\bar{j}}(\mu)} s_{\bar{i}\bar{j}}, \forall i \in N. \tag{A.8}$$

We also have that $v_{\bar{i}}(\mu) \geq \pi_{\bar{i}\bar{j}}(\mu)(s_{\bar{i}\bar{j}} - \delta_{\bar{j}} v_{\bar{j}}(\mu)) + (1 - \pi_{\bar{i}\bar{j}}(\mu))\delta_{\bar{i}} v_{\bar{i}}(\mu)$, and thus

$$\left(1 - (1 - \pi_{\bar{i}\bar{j}}(\mu))\delta_{\bar{i}} + \pi_{\bar{i}\bar{j}}(\mu)\delta_{\bar{j}}\right) \alpha \geq \left(1 - (1 - \pi_{\bar{i}\bar{j}}(\mu))\delta_{\bar{i}}\right) v_{\bar{i}}(\mu) + \pi_{\bar{i}\bar{j}}(\mu)\delta_{\bar{j}} v_{\bar{j}}(\mu) \geq \pi_{\bar{i}\bar{j}}(\mu) s_{\bar{i}\bar{j}},$$

Therefore,

$$\alpha \geq \frac{\pi_{\bar{i}\bar{j}}(\mu)}{1 - \left(1 - \pi_{\bar{i}\bar{j}}(\mu)\right)\delta_{\bar{i}} + \pi_{\bar{i}\bar{j}}(\mu)\delta_{\bar{j}}} s_{\bar{i}\bar{j}} > \frac{\pi_{\bar{i}\bar{j}}(\mu)}{1 + \pi_{\bar{i}\bar{j}}(\mu)} s_{\bar{i}\bar{j}},$$

which contradicts A.8.

Fix a market state μ and a player type i. We next prove that $\mu_i \leq M_i$ implies $\tilde{\mu}_i \leq M_i, \forall \tilde{\mu} \in S(\mu)$. Assume by contradiction that there exist μ, i such that

$$\mu_i \leq M_i$$
$$\mu_i - \sum_{j \in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) + \tilde{\lambda}_i > M_i$$

for some $\tilde{\beta}_{ij} \in \beta_{ij}(\mu) X \left(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu)\right)$ and $\tilde{\lambda}_i \in \lambda_i X(v_i(\mu) - c_i)$. The inequalities above imply that $\tilde{\lambda}_i > M_i - \mu_i + \sum_{j \in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) \geq 0$. Hence $\tilde{\lambda}_i > 0$, which is possible only if

$$v_i(\mu) \geq c_i. \tag{A.9}$$

As $c_i > 0$, the latter inequality implies that

$$\max_{j \in N} s_{ij} > 0. \tag{A.10}$$

Since $\tilde{\lambda}_i \in \lambda_i X(v_i(\mu) - c_i)$, we have $\tilde{\lambda}_i \leq \lambda_i$. Thus $\mu_i - \sum_{j\in N}(\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) + \lambda_i \geq \mu_i - \sum_{j\in N}(\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) + \tilde{\lambda}_i > M_i$, which leads to

$$\mu_i > M_i + \sum_{j\in N}(\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) - \lambda_i \geq M_i - \lambda_i > 0$$

$$\sum_{j\in N}\tilde{\beta}_{ij} \leq \sum_{j\in N}(\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) < \mu_i - M_i + \lambda_i \leq \lambda_i.$$

Therefore,

$$\frac{\sum_{j\in N}\tilde{\beta}_{ij}}{\mu_i} < \frac{\lambda_i}{M_i - \lambda_i}. \tag{A.11}$$

Recall that for $\mu_i > 0$ the payoffs satisfy

$$v_i(\mu) = \sum_{j\in N}\frac{\beta_{ij}(\mu)}{\mu_i}\max(s_{ij} - \delta_j v_j(\mu), \delta_i v_i(\mu)) + \left(1 - \sum_{j\in N}\frac{\beta_{ij}(\mu)}{\mu_i}\right)\delta_i v_i(\mu),$$

which we can rewrite as

$$(1-\delta_i)v_i(\mu) = \sum_{j\in N}\frac{\beta_{ij}(\mu)}{\mu_i}\max(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu), 0).$$

Since $\tilde{\beta}_{ij} \in \beta_{ij}(\mu)X\left(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu)\right)$, we have that

$$\max(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu), 0) \neq 0 \Rightarrow \tilde{\beta}_{ij} = \beta_{ij}(\mu),$$

and hence

$$(1-\delta_i)v_i(\mu) = \sum_{j\in N}\frac{\tilde{\beta}_{ij}}{\mu_i}\max(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu), 0).$$

As $\max(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu), 0) \leq s_{ij}$, we obtain[24]

$$v_i(\mu) \leq \frac{1}{1-\delta_i}\sum_{j\in N}\frac{\tilde{\beta}_{ij}}{\mu_i}s_{ij}. \tag{A.12}$$

Putting together A.9-A.12, we obtain

$$c_i \leq v_i(\mu) \leq \frac{\max_{j\in N} s_{ij}}{1-\delta_i}\sum_{j\in N}\frac{\tilde{\beta}_{ij}}{\mu_i} < \frac{\max_{j\in N} s_{ij}\lambda_i}{(1-\delta_i)(M_i - \lambda_i)}.$$

It follows that

$$M_i < \lambda_i\left(1 + \frac{\max_{j\in N} s_{ij}}{c_i(1-\delta_i)}\right),$$

which contradicts the definition of M_i.

[24]Better upper bounds for $v_i(\mu)$ are achievable, but unnecessary for the argument.

We established that $S(\mu) \subset \mathcal{C}, \forall \mu \in \mathcal{C}$. Therefore, the restriction of S to $\mathcal{C}$ satisfies the hypotheses of Kakutani's fixed point theorem. It follows that S has a fixed point $\mu \in \mathcal{C}$, which constitutes a steady state of the economy with entry costs c. □

Proof of Theorem 3'. The definition of S^ρ and μ^ρ implies that

$$\mu_i^\rho = \rho\left(\mu_i^\rho - \sum_{j\in N}(\tilde{\beta}_{ij}^\rho + \tilde{\beta}_{ji}^\rho)\right) + \tilde{\lambda}_i^\rho,$$
$$\text{for some } \tilde{\beta}_{ij}^\rho \in \beta_{ij}(\mu^\rho)X\left(s_{ij} - \delta_i v_i(\mu^\rho) - \delta_j v_j(\mu^\rho)\right) \text{ and } \tilde{\lambda}_i^\rho \in \lambda_i X\left(v_i(\mu^\rho) - c_i\right).$$

The set $\{(\mu^\rho, \tilde{\beta}^\rho, \tilde{\lambda}^\rho) | \rho \in [0,1]\}$ is contained in a sequentially compact space ($\beta_{ij}(\mu^\rho)$ and $\tilde{\beta}_{ij}^\rho$ are bounded from above because $\mu^\rho \in \mathcal{C}$, $\mathcal{C}$ is compact, and β_{ij} is continuous). Hence there exists a sequence $(\rho^k)_{k\geq 0}$ that approaches 1 such that $(\mu^{\rho^k}, \tilde{\beta}^{\rho^k}, \tilde{\lambda}^{\rho^k})_{k\geq 0}$ converges to some vector $(\mu^1, \tilde{\beta}^1, \tilde{\lambda}^1)$ as $k \to \infty$. The continuity of v and β and the closedness of X's graph imply that

$$\mu_i^1 = \mu_i^1 - \sum_{j\in N}(\tilde{\beta}_{ij}^1 + \tilde{\beta}_{ji}^1) + \tilde{\lambda}_i^1,$$
$$\text{and } \tilde{\beta}_{ij}^1 \in \beta_{ij}(\mu^1)X\left(s_{ij} - \delta_i v_i(\mu^1) - \delta_j v_j(\mu^1)\right) \text{ and } \tilde{\lambda}_i^1 \in \lambda_i X\left(v_i(\mu^1) - c_i\right).$$

Hence μ^1 is a steady state.

We now prove that μ^1 has the desired property. Suppose that $v_i(\mu^1) < c_i$, or equivalently that $X\left(v_i(\mu^1) - c_i\right) = \{0\}$. By continuity, there exists $\underline{k}$ such that $X\left(v_i(\mu^{\rho^k}) - c_i\right) = \{0\}$, so $\tilde{\lambda}_i^{\rho^k} = 0$, for all $k \geq \underline{k}$. By definition,

$$\mu_i^{\rho^k} = \rho^k\left(\mu_i^{\rho^k} - \sum_{j\in N}(\tilde{\beta}_{ij}^{\rho^k} + \tilde{\beta}_{ji}^{\rho^k})\right), \forall k \geq \underline{k}.$$

As $\rho^k < 1$ and $\sum_{j\in N}(\tilde{\beta}_{ij}^{\rho^k} + \tilde{\beta}_{ji}^{\rho^k}) \geq 0$, we need $\mu_i^{\rho^k} = 0$ for all $k \geq \underline{k}$. Therefore, $\mu_i^1 = \lim_{k\to\infty} \mu_i^{\rho^k} = 0$, which completes the proof. □

Proof of Theorem 3". Fix the inflows λ. For every $k > 0$, we need to find entry costs c with $c_i < k$ for all $i \in N$ such that the economy with entry costs c has a positive steady state. As argued in Example 2, for many c, the correspondence S does not have a fixed point that delivers the desired conclusion. We modify the definition of S to obtain a correspondence

$\bar{S} : (0,\infty)^n \rightrightarrows (0,\infty)^n$ as follows

$$\text{(A.13)} \quad \bar{S}(\mu) = \left\{ (\mu_i + \tilde{\lambda}_i - \sum_{j \in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}))_{i \in N} \right|$$

$$\left. \tilde{\beta}_{ij} \in \beta_{ij}(\mu) X\left(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu)\right), \forall i,j \in N \ \& \ \tilde{\lambda}_i \in \lambda_i X(\max(0, v_i(\mu) - k)), \forall i \in N \right\},$$

which differs from S only in the $\tilde{\lambda}$ constraints: the argument $v_i(\mu) - c_i$ has been replaced by $\max(0, v_i(\mu) - k)$.

As in the proof of Theorem 3, we try to restrict the domain and range of $\bar{S}$ to a compact convex subset of $\mathbb{R}^n$. Define

$$\begin{aligned} M_i &= \lambda_i \left(1 + \frac{\max_{j \in N} s_{ij}}{k(1-\delta_i)}\right) \\ \mathcal{C} &= \prod_{i \in N} [\lambda_i, M_i]. \end{aligned}$$

Construct the restriction $\bar{\bar{S}}$ of $\bar{S}$ to $\mathcal{C}$ as follows,

$$\bar{\bar{S}} : \mathcal{C} \rightrightarrows \mathcal{C}, \quad \bar{\bar{S}}(\mu) = \bar{S}(\mu) \cap \mathcal{C}, \forall \mu \in \mathcal{C}.$$

Clearly, $\bar{\bar{S}}$ is convex valued. Arguments similar to those used for Theorem 3 establish that $\bar{\bar{S}}$ has a closed graph on $\mathcal{C}$. In order to apply Kakutani's fixed point theorem, all we have left to argue is that $\bar{\bar{S}}(\mu) \neq \emptyset, \forall \mu \in \mathcal{C}$.

Fix $\mu \in \mathcal{C}$ and $\tilde{\beta}_{ij} \in \beta_{ij}(\mu) X\left(s_{ij} - \delta_i v_i(\mu) - \delta_j v_j(\mu)\right)$ for all $i, j \in N$. For every $i \in N$, we will find $\tilde{\lambda}_i \in \lambda_i X(\max(0, v_i(\mu) - k))$ such that $\mu_i + \tilde{\lambda}_i - \sum_{j \in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) \in [\lambda_i, M_i]$. We consider two cases.

If $v_i(\mu) \leq k$, then $X(\max(0, v_i(\mu) - k)) = [0,1]$, so $\tilde{\lambda}_i$ is constrained to belong to $[0, \lambda_i]$. There exists $\tilde{\lambda}_i \in [0, \lambda_i]$ such that $\mu_i + \tilde{\lambda}_i - \sum_{j \in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) \in [\lambda_i, M_i]$ because

$$\mu_i - \sum_{j \in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) \leq \mu_i \leq M_i$$

$$\mu_i + \lambda_i - \sum_{j \in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) \geq \lambda_i.$$

If $v_i(\mu) > k$, then $X(\max(0, v_i(\mu) - k)) = \{1\}$, which leads to the constraint $\tilde{\lambda}_i = \lambda_i$. We prove that $\mu_i + \lambda_i - \sum_{j \in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) \in [\lambda_i, M_i]$ by contradiction. We clearly have $\mu_i + \lambda_i - \sum_{j \in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) \geq \lambda_i$, so $\mu_i + \lambda_i - \sum_{j \in N} (\tilde{\beta}_{ij} + \tilde{\beta}_{ji}) \notin [\lambda_i, M_i]$ implies that $\mu_i + \lambda_i -$

$\sum_{j\in N}(\tilde{\beta}_{ij}+\tilde{\beta}_{ji}) > M_i$. The latter inequality, along with $v_i(\mu) > k$, leads to a contradiction as in the proof of Theorem 3.

We have verified that $\bar{\bar{S}}$ satisfies the hypotheses of Kakutani's theorem, hence it has a fixed point $\mu \in \mathcal{C}$. One can then easily check that μ is a positive steady state of the economy with entry costs c, where for each $i \in N$, the entry cost c_i is set equal to $v_i(\mu)$ if $v_i(\mu) \leq k$ and can be any arbitrary number in $[0, k]$ if $v_i(\mu) > k$. Basically, in order to make μ a steady state we need to set entry fees so that the players who do not expect payoffs in excess of k are indifferent between entering the market and staying out. □

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