Exchange Rate Risk and Asset Prices

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Abstract

This paper develops a continuous-time model of asset pricing for a global economy with two countries and complete asset markets. In this framework, the stochastic discount factor used to price assets will depend on the currency in which assets' returns are denominated. Using standard continuous-time asset pricing theory, the paper develops the market price of risk equations for domestic and foreign assets denominated in the domestic currency, with currency risk as one of the risk factors driving asset returns, and demonstrates that this risk factor can be represented by the return on the carry trade. A companion paper provides strong empirical evidence for the relevance of the carry trade return as an additional risk factor for U.S. stock returns.

1 Introduction

In this paper, I study the relationship between asset prices and the exchange rate. To study this relationship, I use the tools developed in modern continuous-time asset pricing theory and present them in a framework that is familiar to economists. Standard asset pricing theory is built on the assumption that there are no arbitrage opportunities. If the market is complete and all arbitrage opportunities have been eliminated, there exists a unique risk-neutral probability measure that prices all assets. Expressed in terms of the stochastic discount factor, if arbitrage opportunities are eliminated and markets are complete, there exists a unique stochastic discount factor.

In this paper, I present a model with two countries, domestic and foreign, each with a risky asset and a risk-free bank-account process paying off in the currency of that country. The key ingredient of the model is the stochastic exchange rate that links the two currencies. Asset pricing theory requires that we denominate all assets in a common unit of account, usually the risk-free asset. The unit of account in which assets are denominated is called the numéraire. If the market is complete and all arbitrage opportunities have been eliminated, there exists a unique stochastic discount factor for the financial market. Having two countries leads to two different choices of numéraire. Therefore, there are two different perspectives from which we can analyze the market. For specific I will refer to the two countries as U.S. and Europe.

To make asset prices comparable, American investors use the prevailing exchange rate to express in dollars all European assets that are denominated in Euros. This value substitution adds uncertainty to the payoffs of European assets for American investors, the source of which is currency risk. For American investors, there is only one risk-free asset and consequently, one obvious choice for numéraire, the domestic bank account process. Symmetrically for the European investor uses the European bank account process as numéraire, all American assets are risky. As described by Shreve (2004) and Björk (1998), changing the numéraire changes the risk-neutral probability measure or, equivalently, changes the stochastic discount factor.

In this paper asset prices and the exchange rate are modeled as geometric Brownian motions driven by three factors: one factor for the domestic assets, one factor for the foreign assets, and one factor for the exchange rate. When applying the change of numéraire technique to change from the domestic to the foreign stochastic discount factors, a relationship between the stochastic discount factors emerges. The exchange rate is equal to the ratio of the stochastic discount factors. Backus, Foresi, and Telmer (2001) derived this relationship in a discrete time model with complete markets. For them, as in my model, the stochastic discount factors are different because they price assets denominated in different currencies. Brandt, Cochrane, and Santa-Clara (2006) also acknowledged this relationship between the exchange rate and the stochastic discount factors. However, they identify the domestic or foreign stochastic discount factors with marginal utility growth of the domestic or foreign economies. Consequently, they interpret the difference of stochastic discount factors as evidence of imperfect risk sharing. In the model presented here, different stochastic discount factors arise from different choices of numéraire.

The continuous-time framework allows drawing clear relationships between interest rates, asset prices, and the exchange rate. Uncovered interest-rate parity (UIP) postulates that expected changes in the exchange-rate exactly offset interest rate differentials. According to UIP, the currency from countries with high interest rates should depreciate relative to countries with low interest rates. However, it has been widely recognized that currencies from countries with high interest rates tend to appreciate. This phenomenon is known in the literature as the *forward premium puzzle*. In my framework, an exchange-rate risk premium emerges to explain departures from uncovered interest-rate parity. Brandt and Santa-Clara (2002) presented a continuous-time model of incomplete markets and also derived the foreign exchange rate risk premium. However, in my model, with complete markets, the risk premium has a clearer interpretation; it reflects the relative volatility and the correlation of the stochastic discount factors. It suggests that a positive risk premium can be generated to solve the puzzle by having a domestic stochastic discount factor that is more volatile than the foreign stochastic discount factor or by having a low correlation between the stochastic discount factors. I also show that returns of an asset expressed in different currencies are also bound by the foreign exchange risk premium. The risk premium compensates the domestic investor for all investments that are exposed to exchange-rate risk.

I present conclusions about the exchange-rate risk premium and the relevance of exchange-rate risk as a factor in price assets, in terms of the stochastic discount factors. In the last section of the paper, I present a simple pair-wise estimation of the stochastic discount factors for assets denominated in domestic and foreign currency. The results suggest that foreign exchange risk is a relevant factor to price assets. This risk is best captured by investments in the bank-account processes. In a companion paper, I provide strong empirical evidence of the relevance of exchange-rate risk as an additional factor for U.S. stock returns. In that paper, the return on a portfolio long in currencies with high interest rates and short in currencies in low interest rates, a carry trade, is added to the Fama and French three-factor model for stock returns. The empirical findings indicate that the carry trade return helps price U.S. stock returns.

The rest of this paper is organized as follows: In section 2, I present the theoretical model and its main results. In section 3, I study exchange rate risk. In section 4, I present the empirical results for the pair wise study. Section 5 concludes.

2 Model

There are two countries with their own collections of assets and a stochastic exchange rate that links the two currencies. Financial markets are integrated and can be considered either from the perspective of the domestic investor or from the perspective of the foreign investor. To make asset prices comparable, investors chose a numéraire and express all assets in a common currency. To build the model, I derive the market price of risk equations and solve for the unique stochastic discount factor (SDF), given the choice of numéraire. Finally, I show that the domestic and foreign SDFs are linked by the nominal exchange rate.

2.1 Set up

Consider a model in which asset prices in the world economy are driven by a $D-$dimensional Brownian motion $W = \left(W^1, W^2, ..., W^D\right)$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by W. There are two countries in this economy: domestic and foreign. Each country has its own currency, and assets in each country are denominated in the local currency. There is a nominal exchange rate Q that links the two currencies. Q is itself a stochastic process adapted to the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Assume further that investors can invest freely in any country, there are no arbitrage opportunities, and markets are complete.

There will be $M + N$ assets in the global economy where M is the number of domestic assets and N the number of foreign assets. S_t^{dm} refers to the price at time t of the mth domestic asset with $m = [1, 2, ..., M]$. S_t^{fn} refers to the price at time t of the nth foreign asset with $n = [1, 2, ..., N]$. Subscripts will be used throughout the paper to denote time and superscripts to distinguish assets and countries.

Asset price processes will be modeled as multidimensional geometric Brownian motions, an assumption that is consistent with the limited liability of assets because these processes are guaranteed to be non-negative. Earlier applications of geometric Brownian motions to model asset prices go back to the pioneering work of Merton in continuous-time finance in the late $60's$ and early $70's$ and the revolutionary option-pricing formula of Black and Scholes (1972). This paper adopts a generalization of the Black and Scholes model, in which asset prices are driven by a one-dimensional Brownian motion to a multidimensional framework. The multidimensional assumption is important in the model because I want to distinguish different sources of risk. The components of the $D-$dimensional Brownian motion W will be called the underlying risk factors.

The nominal stochastic price process for the mth domestic asset expressed in differential form is given by

$$dS_t^{dm} = S_t^{dm} \left[\mu_t^{dm} dt + \sum_{j=1}^{D} \sigma_t^{dmj} dW_t^j \right] \tag{1}$$

the parameter μ_t^{dm}, called the drift, can be interpreted as the instantaneous rate of return of the mth domestic asset; the parameter σ_t^{dmj} is interpreted as the instantaneous covariance of the mth domestic asset with the underlying risk factor j. These parameters, which may vary over time, are assumed to be adapted to the filtration generated by the $D-$dimension Brownian motion W. Although equation (1) is driven by a $D-$dimensional Brownian motion, we can interpret equation (1) as a one-dimensional geometric Brownian motion. Let

$$d\mathbb{W}_t^{dm} = \sum_{j=1}^{D} \frac{\sigma_t^{dmj}}{\sigma_t^{dm}} dW_t^j$$

where

$$\left(\sigma_t^{dm}\right)^2 = \sum_{j=1}^{D} \left(\sigma_t^{dmj}\right)^2$$

By Levy's theorem, $\mathbb{W}_t^{dm}$ is a one-dimensional Brownian motion. We regard $\mathbb{W}_t^{dm}$ as the risk factor driving the mth domestic asset, a linear combination of the underlying risk factors W^j. The parameters $\sigma_s^{dmj}/\sigma_s^{dm}$ can be interpreted as the weight of the underlying risk factor W^j on the one-dimensional risk factor $\mathbb{W}_t^{dm}$. Consequently, we can rewrite the stochastic differential equation (1) as a one-dimensional geometric Brownian motion.[1]

$$dS_t^{dm} = S_t^{dm} \left[\mu_t^{dm} dt + \sigma_t^{dm} d\mathbb{W}_t^{m} \right] \tag{2}$$

The parameter σ_t^{dm}, known as the volatility, can be interpreted as the instantaneous standard deviation of the rate of return of the mth domestic asset. Let us further assume, without loss of generality, that one of the assets is

[1] A detailed discussion of Levy's theorem can be found in Shreve (2004) and Proter (2005).

risk-free. This asset is called the domestic bank account. The price process B_t^d represents the cumulative return at time t of a bank account earning the risk-free interest rate and is given by

$$B_t^d = B_0^d \exp\left(\int_0^t r_s^d ds\right) \tag{3}$$

where B_0^d is the initial investment and r_t^d is the instantaneous risk-free interest rate in the domestic country. The price process B^d is strictly positive and deterministic. As is common in finance, I will use the domestic bank account process as the numéraire for domestic economy. The same set-up applies to the foreign country. The foreign nominal asset-price processes are also modeled as geometric Brownian motions. The stochastic-price process for the nth foreign asset is given in differential form by

$$dS_t^{fn} = S_t^{fn}\left[\mu_t^{fn} dt + \sigma_t^{fn} d\mathbb{W}_t^{fn}\right] \tag{4}$$

where

$$d\mathbb{W}_t^{fn} = \sum_{j=1}^{D} \frac{\sigma_s^{fnj}}{\sigma_s^{fn}} dW_s^j$$

and

$$\left(\sigma_t^{fn}\right)^2 = \sum_{j=1}^{D} \left(\sigma_t^{fnj}\right)^2$$

Consequently, $\mathbb{W}^{fn}$, which is regarded as the risk factor driving the nth foreign asset, is a one-dimensional Brownian motion. The drift μ^{fn} can be interpreted as the instantaneous rate of return of the nth asset in foreign currency and the volatility σ^{fn} as the instantaneous volatility of the nth foreign asset. As in the domestic economy, I will assume that one of the foreign assets is the risk-free bank-account process B^f that serves as numéraire in the foreign country and whose price at time t is given by

$$B_t^f = B_0^f \exp\left(\int_0^t r_s^f ds\right) \tag{5}$$

where B_0^f is the initial investment and r_t^f is the instantaneous risk free rate in the foreign country.

Thus far, I have presented each country separately, each one with its own collection of risky assets and numéraire denominated in the local currency.

In the familiar framework in finance, a country is analyzed according to its own collection of assets that are denominated in the local currency. However, in international finance, there are more than one currency, and investment opportunities include foreign assets. To be comparable, nominal assets need to be expressed in the same currency. Hence, there is more than one choice of numéraire. To price assets in a common currency, I introduce the nominal exchange rate Q, defined as the number of domestic currency units required to purchase one unit of foreign currency. The exchange rate is the most important piece of the model. It links the two currencies and adds uncertainty to returns on investments made in a different currency. The same asset, when looked at from the perspective of a domestic agent, is different when it is looked at from the perspective of a foreign agent. Formally, the nominal exchange-rate process is assumed to be a geometric Brownian motion adapted to the filtration generated by the same $D-$dimensional Brownian motion W that drives the risky assets. The stochastic process for the nominal exchange rate in differential form is given by

$$dQ_t = Q_t \left[\mu_t^Q dt + \sigma_t^Q d\mathbb{W}_t^Q \right] \tag{6}$$

where

$$d\mathbb{W}_t^Q = \sum_{j=1}^{D} \frac{\sigma_s^{Qj}}{\sigma_s^Q} dW_s^j$$

and

$$\left(\sigma_t^Q\right)^2 = \sum_{j=1}^{D} \left(\sigma_t^{Qj}\right)^2$$

With this characterization of the nominal exchange-rate process, we can interpret $\mathbb{W}_t^Q$ as the exchange-rate risk factor. The drift μ_t^Q is the instantaneous expected rate of depreciation/appreciation of the domestic currency, and the volatility parameter σ_t^Q is the instantaneous standard deviation of the rate of depreciation/appreciation of the domestic currency.

2.2 Exchange Rate and Asset Prices

With this description of the assets available in each country, as well as the exchange rate dynamics, we can express all assets in a common currency, either domestic or foreign. An investor's asset allocation depends on the currency that is used to express payoffs, and the relevant currency to price assets for an investor is the one from the country in which the investor lives. Throughout this exposition, I will adopt the perspective of a domestic investor, an agent who lives in the domestic country and cares about payoffs in the domestic currency. Because of the symmetry of the model, the same results would hold if we were to adopt the perspective of a foreign investor.

The goal in this section is to analyze the two countries in the integrated global financial market. By expressing all assets in the same currency, we can study the financial market from the perspective of a specific country. We will show that asset-price processes denominated in different currencies are affected by the exchange rate; consequently, the market looks different for agents in different countries. We will also show that there is only one asset that is risk-free for each investor. With the continuous-time framework, this result can be shown simply and clearly.

The product of two geometric Brownian motions is itself a geometric Brownian motion. Thus, asset-price processes that are geometric Brownian motions in one currency remain geometric Brownian motions when translated to the other currency. Formally, for any asset n that is denominated in the foreign currency, the price process in the domestic currency is given by $\left(Q_t S_t^{fn}\right)_{t\geq 0}$. By Ito's product rule, the stochastic price process of a foreign asset in domestic currency, in differential form, is given by

$$d\left(Q_t S_t^{fn}\right) = Q_t S_t^{fn}\left[\left(\mu_t^{fn} + \mu_t^{Q} + \sigma_t^{fn}\sigma_t^{Q}\rho_t^{fnQ}\right)dt + \sigma_t^{fn} d\mathbb{W}_t^{fn} + \sigma_t^{Q} d\mathbb{W}_t^{Q}\right] \tag{7}$$

where

$$\rho_t^{fnQ} = \frac{\sum_{j=1}^{D}\sigma_t^{Qj}\sigma_t^{fnj}}{\sigma_t^{fn}\sigma_t^{Q}}$$

ρ_t^{fnQ} is the instantaneous correlation between the exchange rate and the foreign asset and $\sigma_t^{fn}\sigma_t^{Q}\rho_t^{fnQ}$ is interpreted as the instantaneous co-variation between the foreign asset and the exchange rate.[2] Equation (7) has the form of a multidimensional geometric Brownian motion. The drift, which can be interpreted as the instantaneous expected rate of return in the domestic currency of a foreign asset, is equal to the sum of three components: the expected return of the foreign asset denominated in the foreign currency μ_t^{fn}, the expected depreciation/appreciation of the currency μ_t^{Q}, and the instantaneous

[2] See Shreve (2004) for a discussion of instantaneous correlation.

co-variation between the exchange rate and the foreign asset $\sigma_t^{fn}\sigma_t^{Q}\rho_t^{fnQ}$. The components of W that drive the price processes in domestic currency are the risk factor $\mathbb{W}^{fn}$ that drives the foreign stock and the risk factor $\mathbb{W}^{Q}$ that drives the foreign exchange. Thus, all foreign assets denominated in domestic currency are affected by the exchange-rate risk factor. Comparing equation (7) to equation (4) , which I repeat here for convenience,

$$dS_t^{fn} = S_t^{fn}\left[\mu_t^{fn}dt + \sigma_t^{fn}d\mathbb{W}_t^{fn}\right]$$

we see that, when expressed in its own currency, the drift of the foreign asset price process is simply μ_t^{fn}, but when translated into a new currency, the drift also depends on the expected rate of appreciation/depreciation of the exchange rate and the correlation between the exchange rate and the foreign asset.

If there are no arbitrage opportunities, there can be only one risk-free asset in the model. Although there are two bank account processes in this model, one for each country, only one of these processes is risk-free when assets are denominated in a common currency. If this were not the case, there would be an opportunity for arbitrage. When an investment is made in the foreign bank account, the price process of this investment denominated in the domestic currency is given by

$$d\left(Q_tB_t^f\right) = Q_tB_t^f\left[\left(\mu_t^Q + r_t^f\right)dt + \sigma_t^Q d\mathbb{W}_t^Q\right] \tag{8}$$

.It is clear from equation (8) that an investment in the foreign bank account process is not risk free for a resident of the domestic economy because of exchange-rate risk. The same is true, of course, for investments by foreign agents in the domestic bank account process. For any investor in this model, only the risk-free asset from the country in which the investor is resident is risk free; all other assets are risky.

To make assets prices comparable, we express asset prices in the same currency. Let P_t^{dk}, $k = [1, 2, ..., M + N]$ be the price of an asset at time t denominated in the domestic currency. Foreign asset prices are converted to the domestic currency by multiplying by the exchange rate; then, we have

$$P_t^{dk} = \begin{cases} S_t^{dm} & for\ k = 1, 2, .., M,\ m = k \\ Q_tS_t^{fn} & for\ k = M + 1, ..., M + N,\ n = k - M \end{cases}$$

that is, for $k \leq M$, we refer to a domestic asset, and for $k \geq M+1$, we refer to a foreign asset. P_t^d is the collection of asset-price processes denominated in the domestic currency. From equations (2) and (7), the stochastic equation for P^{dk} is given in differential form by

$$dP_t^{dk} = P_t^{dk}\left[\mu_t^{dm}dt + \sigma_t^{dm}d\mathbb{W}_t^{dm}\right] \tag{9a}$$

for $k = [1, 2, ..., M]$, $m = k$ and

$$dP_t^{dk} = P_t^{dk}\left[\left(\mu_t^{fn} + \mu_t^{Q} + \sigma_t^{fn}\sigma_t^{Q}\rho_t^{Qfn}\right)dt + \sigma_t^{fn}d\mathbb{W}_t^{fn} + \sigma_t^{Q}d\mathbb{W}_t^{Q}\right] \tag{9b}$$

for $k = [M+1, M+2, ...M+N]$, $m = k - M$

Furthermore, let P_t^{fk} be the price of an asset at time t, denominated in the foreign currency where

$$P_t^{fk} = \begin{cases} S_t^{dm}/Q_t & for\ k = 1, 2, .., M,\ m = k \\ S_t^{fn} & for\ k = M+1, ..., M+N,\ n = k - M \end{cases}$$

and let P_t^f be the collection of asset-price processes denominated in the foreign currency. Clearly, $P^{fk} = QP^{dk}$ for all k.

For the purpose of this paper, s financial market is a collection of asset-price processes P, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. There is one integrated financial market in this model that can be analyzed either from the perspective of the domestic investor, with the price processes denominated in the domestic currency P^d, or from the perspective of the foreign economy, with the price processes denominated in foreign currency P^f.

2.3 The Stochastic Discount Factor

Modern asset pricing theory relies on the assumption of no arbitrage. Although the assumption of no arbitrage is enough to yield powerful conclusions, it is a weak assumption relative to those usually imposed in economics. As long as there is a financial market that does not admit arbitrage, we do not need to include assumptions about agent's preferences, aggregation, or how expectations are formed. The absence of arbitrage opportunities is enough to yield a model of how assets are priced.

The basic concept of asset-pricing theory is that discounted asset prices are martingales under an equivalent martingale measure (EMM), also called the risk-neutral probability measure. This condition is known as the fundamental equation of asset pricing. The first fundamental theorem of asset pricing links no arbitrage to the fundamental equation. It states that in a financial market in the absence of arbitrage, there exists a risk-neutral probability measure. That is, in the absence of arbitrage opportunities, we can find a probability measure $\widetilde{\mathbb{P}}$, such that the discounted asset-price processes are martingales; for every time interval $\Delta > 0$

$$\widehat{P}_t = \widetilde{E}\left[\widehat{P}_{t+\Delta} \left| \mathcal{F}_t \right.\right] \tag{10}$$

where $\widehat{P}_t = P_t/B_t$ is the discounted asset-price process. $\widehat{P}_t$ expresses the nominal price of the asset in terms of the numéraire B. The expectation $\widetilde{E}$ in (10) is taken under the risk-neutral probability measure $\widetilde{\mathbb{P}}$. In continuous time, under certain technical conditions that I will assume are satisfied, Girsanov's theorem provides a way to construct a EMM.[3] In fact, under the assumption that markets are complete, the second fundamental theorem of asset pricing guarantees that the EMM is unique.

Instead of changing the true probability measure to the risk-neutral probability measure, economists like to express the fundamental equation of asset pricing in terms of the stochastic discount factor. Intuitively, the SDF is the random process that agents use to discount future payoffs. Agents value an asset depending on the circumstances of its payoffs; for example, assets that pay in a few bad states but do not pay in many good states are valuable because the asset provides a hedge. Agents are willing to pay insurance even if, on average, it produces losses. The SDF weights payoffs depending on the state, discounting payoffs in good states more heavily then payoffs in bad states. Formally, the SDF is a stochastic process Λ, such that, under the true probability measure $\mathbb{P}$, the $SDF-$weighted asset prices are martingales; that is,

$$\Lambda_t P_t = E\left[\Lambda_{t+\Delta} P_{t+\Delta} \left| \mathcal{F}_t \right.\right]$$

Existence of an EMM and existence of a SDF are equivalent. Cochrane (2005) advocated a SDF approach to asset-pricing theory in an effort to bring finance and economics together. Therefore, this paper will also opt for a SDF approach. I now turn to a derivation of the SDF for the domestic and foreign economies.

Consider the financial market from the perspective of the domestic investor. The relevant price processes and choice of numéraire to derive the SDF in the domestic economy are P^d and B^d respectively. Assume that

[3] For a detailed discussion of Girsanov's theorem see Shreve (2004) and Protter (2005)

the adapted process $\theta^{d1}, \theta^{d2}, ..., \theta^{dD}$ satisfies the conditions of Girsanov's theorem. Then the stochastic discount factor in the domestic economy is the strictly positive Ito process Λ^d, given, in differential form, by

$$d\Lambda_t^d = \Lambda_t^d \left[-r_t^d dt - \sum_{j=1t}^{D} \theta^{dj} dW_t^j \right] \tag{11}$$

The volatility parameters $\theta^{d1}, \theta^{d2}, ..., \theta^{dD}$ are the market prices of risk, in the domestic economy, for the underlying risk factors W^j, $j = [1, 2, ..., D]$. Recall from the fundamental equation of asset pricing that the SDF for the domestic economy is the process such that the $SDF-$weighted asset prices denominated in the domestic currency are martingales. Then for any asset k denominated in the domestic currency, the drift of the process $\left(\Lambda_t^d P_t^{dk}\right)_{t\geq 0}$ must equal zero. Therefore, if P_t^{dk} refers to a domestic asset, $\left(\Lambda_t^d P_t^{dk}\right)$ is a martingale if

$$\mu_t^{dm} - r_t^d = \sum_{j=1}^{D} \sigma_t^{dmj} \theta_t^{dj} \tag{12a}$$

.If P_t^{dk} refers to a foreign asset $\left(\Lambda_t^d P_t^{dk}\right)$ is a martingale if

$$\mu_t^{fn} + \mu_t^Q + \sigma_t^{fn} \sigma_t^Q \rho_t^{fnQ} - r_t^d = \sum_{j=1}^{D} \left(\sigma_t^{fnj} + \sigma_t^{Qj}\right) \theta_t^{dj} \tag{12b}$$

We call this equation the domestic market price of risk equation for asset k.[4] This equation states that the expected excess return of asset k in the domestic currency is equal to the sum of prices of risk times quantity of risk, summed over the D types of risk. This justifies calling θ^{dj} a domestic price of risk.

When the asset is domestic, equation $(12a)$ is the standard market price of risk equation that is derived in a financial market with just one currency.[5] When the asset is foreign, equation $(12b)$ is the corresponding domestic market price of risk equation. What is distinctive for foreign assets is that the domestic market price of risk equation includes extra parameters. These parameters correct the market price of risk equation for exchange rate risk. On the left hand side of equation $(12b)$, the excess return is corrected by

[4] See the appendix for the derivation of these prices of risk equations.

[5] Chapter 5 in Shreve (2004) discusses the market price of risk equation.

adding the drift of the exchange-rate process μ_t^Q and the co-variation of the foreign asset with the exchange rate $\sigma_t^{fn}\sigma_t^Q\rho_t^{fnQ}$. On the right hand side, the correlation of the asset with the underlying risk factors is corrected by adding the correlation of the exchange-rate process with the underlying risk factors. Equation (12b) is what distinguishes models in international finance.

The market price of risk equations does not rely on the assumption of complete markets. These equations should be satisfied in either a complete or an incomplete market framework. The assumption that markets are complete guarantees that, from the collection of market price of risk equations, we can solve for the unique domestic market prices of risk θ_t^{dj}. On the other hand, if markets are not complete, there is not a unique solution for the domestic market prices of risk. When markets are not complete, the SDF is not unique. In this paper, we assume that markets are complete. Formally, on the global financial market, there is a $D-$dimensional Brownian motion driving $M+N$ assets. Under the assumption that markets are complete, we can, without loss of generality, assume that $D = M+N-1$ so that we have as many risky assets as underlying risk factors.[6] With complete markets, there are D domestic market price of risk equations in the D unknown processes $\theta^{d1}, \theta^{d2}, ..., \theta^{dD}$. Therefore, as long as the matrix of volatilities in the local currency Σ^d is invertible, where

$$\Sigma^d = \begin{bmatrix} \sigma^{d11} & \sigma^{d12} & \dots & \sigma^{d1D} \\ \sigma^{d21} & \sigma^{d22} & & \sigma^{d2D} \\ \vdots & \vdots & & \vdots \\ \sigma^{dN1} & \sigma^{dN2} & \dots & \sigma^{dND} \\ \sigma^{f11}+\sigma^{Q1} & \sigma^{f12}+\sigma^{Q2} & \dots & \sigma^{f1D}+\sigma^{QD} \\ \vdots & \vdots & & \vdots \\ \sigma^{fM1}+\sigma^{Q1} & \sigma^{fM2}+\sigma^{Q2} & \dots & \sigma^{fMD}+\sigma^{QD} \end{bmatrix}$$

we can solve for the unique domestic market prices of risks.

I have shown that, in the absence of arbitrage opportunities and with complete markets, when we look at the financial market from the perspective of the domestic investor, there is a unique process that prices all assets when denominated in the domestic currency. We can also look at this financial market from a different perspective, that of the foreign investor. I will show

[6]If we allow more risky assets than underlying risk factors, we need to impose cross-equation restrictions to satisfy the no-arbitrage assumption.

that the foreign SDF process that prices assets in the foreign currency is different from the domestic SDF process that prices assets in the domestic currency.

Let us consider the financial market from the perspective of the foreign investor. Following the same arguments for the derivation of the SDF in the domestic economy, we derive the SDF in the foreign economy. In this case, the relevant price processes and choice of numéraire to derive the SDF in the foreign economy are P^f and B^f, respectively. Under the assumption that markets are complete, the matrix of volatilities in the foreign currency Σ^f is invertible. The unique stochastic discount factor for the foreign economy is the Ito process Λ^f, given in differential form by

$$d\Lambda_t^f = \Lambda_t^f \left[-r_t^f dt - \sum_{j=1}^{D} \theta_t^{fj} dW_t^j \right] \tag{13}$$

where θ^{fj} stands for the foreign price of risk for factor j.

The assumptions of the model, namely, no arbitrage and complete markets, guarantee a unique SDF for the domestic economy Λ^d and a unique SDF for the foreign economy Λ^f. These two stochastic processes are driven by the same underlying risk factors W. However, the parameters for these two processes are different: The drift on each process is equal to minus the risk-free interest rate in the local economy. Volatilities are the market prices of risks that solve the system of the market prices of risk equations in the local economy. Although different, in this integrated financial market, the parameters of these two processes are connected by the exchange rate. In the next section, I study this link.

2.4 Stochastic Discount Factors and the Exchange Rate

Up to this point, we have solved for the stochastic discount factors for each country separately. We used the asset price processes P^d and numéraire B^d to solve for the SDF in the domestic economy and the asset price processes P^f and numéraire B^f to solve for the SDF in the foreign economy. The exchange rate is what links the two countries in the integrated financial market. Here, I show how the exchange rate process Q_t links the stochastic discount factors processes Λ^d and Λ^f. To gain some insight into the role of the exchange rate in the model, I first consider the case in which the exchange-rate process is deterministic, before moving on to the more interesting case in which the exchange rate is stochastic. I will show that, when the exchange rate is deterministic, the market prices of risk are equal in the two economies. On the other hand, when the exchange rate is stochastic, the market prices of risk are different between the two economies but connected by the exchange rate.

I begin with the simple and unrealistic assumption that the exchange rate process is deterministic and given by $dQ_t = Q_t \mu_t^Q dt$. I will show that, for investment purposes, the financial market looks equally risky from the perspective of the domestic and the foreign investor. Consider the price process for a foreign asset denominated in the local currency when the exchange rate is deterministic, that is

$$d\left(Q_t S_t^{fn}\right) = Q_t S_t^{fn} \left[\left(\mu_t^{fn} + \mu_t^Q\right) dt + \sigma_t^{fn} d\mathbb{W}_t^{fn}\right] \tag{14}$$

Equation (14) corresponds to equation (7) with $\sigma^Q = 0$. Comparing these two equations, we see that, when the exchange rate is stochastic, there are two risk factors driving the price process of a foreign asset denominated in the local currency: the foreign-stock risk factor $\mathbb{W}^{fn}$ and the foreign-exchange risk factor $\mathbb{W}^Q$. When the exchange rate is deterministic, the latter term is lost, and foreign asset prices denominated in the domestic currency are driven only by the foreign-stock risk factor $\mathbb{W}^{fn}$. Consequently, assets are equally risky from the perspective of both domestic and foreign investors. In this simple framework of deterministic exchange rates, there are two risk-free assets in the economy: the domestic bank-account process and the foreign bank-account process. To be consistent with the hypothesis of an absence of arbitrage opportunities, we require the expected depreciation/appreciation of the currency to be equal to the interest rate differential, that is, $\mu_t^Q = r_t^d - r_t^f$. Otherwise, there will be two risk-free assets with different risk-free returns in the model, providing an opportunity for an arbitrage. When the exchange rate is deterministic, it is easy to show that the market price of risk equations are identical for both domestic and foreign investors. For a domestic asset, the market price of risk equation is from either perspective

$$\mu_t^{dm} - r_t^d = \sum_{j=1}^{D} \sigma_t^{dmj} \theta_t^j$$

for $m = [1, 2, ..., M]$. For a foreign asset, the market price of risk equation is

$$\mu^{fn} - r_t^f = \sum_{j=1}^{D} \sigma_t^{fnj} \theta_t^j$$

for $n = [1, 2, ..., N]$. Because the market prices of risk equations are identical for both domestic and foreign investors, prices of risk are identical for both domestic and foreign investors. Note that I dropped the superscript that refers to the country from the prices of risk in order to emphasize that they are equal.

When there is no volatility on the exchange rate process, we can analyze the financial market from the perspective of either a domestic or a foreign investor and arrive at the same conclusion. In this context, market prices of risk are global. On the other hand, from the standpoint of this paper, when the exchange rate is stochastic, the investor's residence matters.

Now we turn our attention to the more interesting case in which the exchange rate is stochastic. I will show how the exchange rate links the two economies in my model. Intuitively, the exchange rate adds risk for investing in non-local assets; as a result, price processes for foreign and domestic investors are different. Therefore, market prices of risk are different across countries. We should expect, then, that the exchange rate reflects those differences in the market prices of price risk.

Proposition 1 *Let Λ^d be the SDF for the domestic economy, Λ^f the SDF for the foreign economy, and Q the exchange rate process. Then the following relation holds $\Lambda_t^f = \Lambda_t^d Q_t / Q_0$ for all $t \geq 0$, and*

$$\mu_t^Q = r_t^d - r_t^f + \beta_t^d \tag{15}$$

where

$$\beta_t^d = \sum_{j=1}^{D} \theta_t^{dj} \left(\theta_t^{dj} - \theta_t^{fj} \right) \tag{16}$$

and the volatility of the exchange rate process satisfies

$$\left(\sigma_t^Q \right)^2 = \sum_{j=1}^{D} \left(\theta_t^{dj} - \theta_t^{fj} \right)^2 \tag{17}$$

for all $t \geq 0$.[7]

[7] The proof is in the appendix

The relationship between the stochastic discount factors and the exchange rate $\Lambda_t^f = \Lambda_t^d Q_t/Q_0$ is equivalent to the discrete time derivation of Backus, Foresi, and Telmer (2001). Equation (15) for the drift of the exchange rate process is the continuous-time analog of a familiar result in the international finance literature: the expected appreciation/depreciation of the currency is equal to the interest-rate differential plus a risk premium. This is the continuous-time analog of the uncovered interest-rate parity condition. In this model, the risk premium is the weighted sum of the difference in the prices of risks between the two SDF, where the domestic prices of risk are used to weight the differences. I refer to the risk premium β_t^d as the foreign-exchange risk premium for the domestic economy. The instantaneous volatility of the exchange-rate process in equation (17) is the Euclidean norm of the difference on the prices of risk, that is $\sigma_t^Q = \left\|\theta_t^d - \theta_t^f\right\|$. The volatility of the exchange rate reflects the difference in the prices of risk for the two economies. Therefore, a stochastic exchange rate implies that the numéraire-dependent SDFs are different.

Just as the stochastic discount factors are linked by the exchange rate, the expected excess returns for an asset, when it is denominated in different currencies, are also linked by the exchange rate. To see this result, note that, from the drift of the exchange-rate process in equation (15) and the left-hand side of the domestic market price of risk equation for a foreign asset in equation $(12b)$, we can rewrite the instantaneous expected excess return for a foreign asset denominated in the domestic currency as

$$\mu_t^{fn} + \mu_t^Q + \sigma_t^{fn}\sigma_t^Q\rho_t^{fnQ} - r_t^d \equiv \mu_t^{fn} - r_t^f + \beta_t^d + \sigma_t^{fn}\sigma_t^Q\rho_t^{fnQ}$$

The expected excess return for the foreign asset denominated in the domestic currency is equal to the expected excess return of the foreign asset denominated in the foreign currency $\mu_t^{fn} - r_t^f$ plus a risk premium. The risk premium has two components: The first component is the foreign exchange risk premium for the domestic economy β_t^d; this component of the premium is independent of the asset, justifying calling β_t^d the foreign exchange risk premium for the domestic economy. The second component of the premium is an asset-specific premium, which is equal to the co-variation of the foreign asset with the foreign exchange $\sigma_t^{fn}\sigma_t^Q\rho_t^{fnQ}$.

In an integrated financial market, the exchange rate links the numéraire-dependent stochastic discount factors. Both the drift and the volatility of the exchange rate capture the difference in the market prices of risk for the two economies. These differences in the numéraire-dependent SDF lead to premiums for investing in foreign assets. One component of the premium is independent of the asset β_t^d while the other component is asset dependent. In the next sections, I take a closer look at the foreign exchange risk premium and study its relevance for a domestic investor.

2.5 Foreign-Exchange Risk Premium

The must interesting parameter in the above derivation is the foreign-exchange risk premium for the domestic economy β_t^d. In our model, the foreign-exchange risk premium arises because assets pay in money and different countries denominate assets in different currencies. In this section, I examine how the foreign-exchange risk premium relates to the numéraire-dependent stochastic discount factors and show that the foreign-exchange risk premium depends on the country of reference.

To simplify this exposition, I express the domestic and foreign SDF as a one-dimensional geometric Brownian motion. Let

$$dW_t^{\Lambda^d} = \sum_{j=1}^{D} \frac{\theta_t^{dj}}{\theta_t^d} W_t^j$$

and

$$\left(\theta_t^d\right)^2 = \sum_{j=1}^{D} \left(\theta_t^{dj}\right)^2 \tag{18}$$

By Levy's theorem, $W_t^{\Lambda^d}$ is a one-dimensional Brownian motion, and consequently, we can rewrite the differential for the domestic SDF in equation (11) as

$$d\Lambda_t^d = \Lambda_t^d \left[-r_t^d dt - \theta_t^d dW_t^{\Lambda^d}\right]$$

,where the volatility parameter θ_t^d is interpreted as the instantaneous standard deviation of the rate of growth of the domestic SDF. Similarly, the foreign SDF can be expressed as a one-dimensional Brownian motion with volatility θ_t^f,

$$d\Lambda_t^f = \Lambda_t^f \left[-r_t^f dt - \theta_t^f dW_t^{\Lambda^f}\right]$$

where

$$dW_t^{\Lambda^f} = \sum_{j=1}^{D} \frac{\theta_t^{fj}}{\theta_t^f} W_t^j$$

and

$$\left(\theta_t^f\right)^2 = \sum_{j=1}^{D} \left(\theta_t^{fj}\right)^2 \tag{19}$$

Then, we can express the foreign-exchange risk premium for the domestic economy in terms of the volatilities of the numéraire-dependent SDFs and their correlation.[8]

$$\beta_t^d = \theta_t^d \theta_t^f \left(\frac{\theta_t^d}{\theta_t^f} - \rho_t^{f,d}\right) \tag{20}$$

where $\rho_t^{f,d}$ is the correlation between the domestic and foreign SDF. The sign for the foreign-exchange risk premium comes from two components: the relative volatilities of the domestic and foreign stochastic discount factors and the correlation between these two processes. A strong positive correlation lowers the foreign-exchange risk premium while higher volatility at home increases the foreign-exchange risk premium for the domestic economy. In fact, if the ratio of volatilities of the SDFs is equal to their correlation, the foreign-exchange risk premium is zero, even though the numéraire-dependent SDFs could be different. That is, a stochastic exchange rate that necessarily implies different SDFs for the domestic and foreign economy, does not necessarily imply a foreign-exchange risk premium for the domestic economy.

By the symmetry of the model, the foreign exchange risk premium for the foreign economy is

$$\beta_t^f = \theta_t^d \theta_t^f \left(\frac{\theta_t^f}{\theta_t^d} - \rho_t^{f,d}\right)$$

Thus, the absence of a foreign-exchange risk premium in the domestic economy does not imply the absence of a foreign-exchange risk premium in the foreign economy unless, of course, the correlation is equal to 1 and the volatilities of the SDFs are the same $\theta_t^f = \theta_t^d$. In fact, little can be said about the foreign-exchange risk premium for the foreign country by simply observing the foreign-exchange risk premium in the domestic economy. This observation is relevant because it implies that the discussion of the foreign-exchange risk premium depends on the country of reference. For this discussion, we adopt the perspective of the domestic investor. In the next section, I investigate how currency risk co-varies with the domestic SDF.

[8] See the appendix for this derivation.

3 Foreign Exchange Risk

For a domestic investor, the most direct way to hedge for foreign exchange risk is to invest in the foreign bank-account process. The only risk factor driving this asset is the foreign-exchange risk factor W_t^Q. In this section, I investigate the relationship between the foreign-exchange risk and the domestic SDF by examining the relationship between the price processes for investing in the foreign bank account and the domestic SDF. If there is a strong correlation between these two processes, then empirically, currency risk might be a relevant factor in price stock returns.

Linking the results from the previous section, we can rewrite the differential of the price process for investing in the foreign bank-account process, denominated in the domestic currency as

$$d\left(Q_t B_t^f\right) = Q_t B_t^f \left[\left[r_t^d + \beta_t^d\right] dt + \sigma_t^Q dW_t^Q\right] \tag{21}$$

The instantaneous expected return for this investment is equal to the sum of two components: the domestic risk-free rate r^d and the foreign-exchange risk premium for the domestic economy β_t^d. As discussed in the previous section, the foreign-exchange risk premium can be either positive or negative. Thus, according to the sign of the risk premium, the instantaneous expected return for investing in the foreign bank-account process could be higher or lower than the domestic risk-free interest rate.

To determine how the return on the investment in the foreign bank-account process moves with the domestic stochastic discount factor, we examine the quadratic co-variation process for the log foreign bank-account process in domestic currency $\ln QB^f$ and the log domestic SDF $\ln \Lambda^d$.[9] The differential of the quadratic co-variation process is given by

$$d\left[\ln\left(QB^f\right), \ln \Lambda^d\right]_t = -\sum_{j=1t}^{D} \theta_t^{dj}\left[\theta_t^{dj} - \theta_t^{fj}\right] dt = -\beta_t^d dt \tag{22}$$

The quadratic co-variation is informally regarded as the instantaneous covariation of the return on the foreign bank-account process and the growth rate of the domestic SDF.

$$\frac{d\Lambda_t^d}{\Lambda_t^d} \frac{d\left(Q_t B_t^f\right)}{Q_t B_t^f} = -\sum_{j=1}^{D} \theta_t^{dj}\left[\theta_t^{dj} - \theta_t^{fj}\right] dt = -\beta_t^d dt$$

[9] See Protter (2005) for a formal discussion of quadratic co-variation.

The co-variation between these two processes is equal to the negative of the foreign-exchange risk premium for the domestic economy β_t^d. Recall that the SDF weights future payoffs depending on the state. A negative co-variation between the log of the domestic SDF and the return of an asset, indicating that the asset is more likely to pay in good states than in bad states; in this sense, the asset is very risky. On the other hand, if the co-variation is positive, then the asset is more likely to pay in bad states than in good states, and the asset provides a hedge. Note that when I refer to *how risky* an asset is, I do not refer to higher or lower volatility of the returns. Except for the risk-free asset in the local country, all other assets are risky for the domestic investor because their returns are stochastic. When I refer to an asset as very risky, it is because the co-variation of the return of the asset with the log of the domestic SDF is negative, and when I refer to an asset as one that provides a hedge, it is because the co-variation of the return of the asset with the log of the domestic SDF is positive.

Equations (21) and (22) link the foreign exchange risk premium for the domestic economy with the instantaneous co-variation between the return for investing in the foreign bank-account process and the domestic SDF. *The foreign exchange risk premium is equal to the negative of the co-variation between the log domestic SDF and the returns in domestic currency for investing in the foreign bank account process.* I now analyze this result in the context of the model. We consider three cases:

1. $\beta^d = 0$ (There is no foreign exchange risk premium). The expected return for investing in the foreign bank account process is equal to the domestic risk-free interest rate, and the co-variation between this return and the log domestic SDF equals zero. Here, the uncovered interest-rate parity holds; that is, the expected appreciation/depreciation of the currency is equal to the risk-free interest rate differential. Domestic investors are not compensated for being exposed to currency risk, and the exchange-rate risk is not correlated with the domestic SDF. The extreme case of this scenario corresponds to a deterministic exchange rate illustrated in section 2.4. When the exchange rate is deterministic, there is no exchange-rate risk and, thus, no premium for investing in foreign assets.

2. $\beta^d > 0$ (The foreign exchange risk premium is positive). In this case, the expected return for investing in the foreign bank-account process is

greater than the domestic risk-free interest rate, and the co-variation between this return and the log domestic SDF is negative. This is the case of a very risky asset. When the macro state of the economy is bad, the payoff for investing in the foreign bank-account process is bad, and domestic investors demand to be compensated for being exposed to exchange-rate risk. Consequently, the expected return for investing in the foreign bank-account process is higher than the rate of return of the domestic risk-free asset. A positive foreign-exchange risk premium is usually found in the data.

3. $\beta^d < 0$ (The risk premium is negative).Finally, consider the case in which the expected return for investing in the foreign bank-account process is lower than the domestic risk-free interest rate, and the co-variation between this return and the log domestic SDF is positive. In this case, a long position in the foreign bank-account process provides a hedge against bad states in the domestic economy; the payoff is good in bad times and not so good in good times. Holding the asset provides insurance, so investors are willing to accept an expected return lower than the risk-free interest rate.

If there is a positive foreign exchange risk premium, as documented in the data, then investments in the foreign bank-account process are risky bets. We can also express this relationship in terms of the instantaneous correlation between the domestic SDF an the foreign exchange risk. The correlation tell us how strong is the relationship between these two processes. The instantaneous correlation between the domestic SDF and the exchange-rate risk is given by

$$\rho_t^{\Lambda^d Q} = \frac{\beta_t^d}{\theta_t^d \sigma_t^Q}$$

We can express this equation in terms of the volatilities of the domestic and foreign SDFs and their correlation, exactly as we did for the foreign exchange risk premium in (20):

$$\rho_t^{\Lambda^d Q} = \frac{\left(\theta_t^d - \theta_t^f \rho_t^{f,d}\right)}{\sqrt{\left(\theta_t^d\right)^2 + \left(\theta_t^d\right)^2 - 2\theta_t^d \theta_t^f \rho_t^{f,d}}} \tag{23}$$

Clearly, without looking at data, we cannot say whether the exchange-rate risk is correlated with the domestic SDF. In the next section I implement the model to get a sense of the extent in which foreign exchange risk is correlated with the domestic SDF.

4 Pair-wise Study

In this section, I implement the continuous-time model presented above in a pair-wise empirical study. I consider a pair of countries each with one stock and a risk-free asset denominated in the local currency. We begin with the assets to derive the domestic and foreign SDFs. Then, I investigate the relationship between the domestic SDF and the exchange rate risk. I compare six pairs of countries, in which the U.S. is always considered the domestic country. The foreign countries are Australia, Canada, Japan, Switzerland, the UK, and the Euro zone.

To adapt the model to the data, we need to go from the continuous-time model to a daily model. In this section, I assume that all the parameters in the model are fixed. For any asset, the price process paying off in dollars, P_t^{dk}, is a geometric Brownian motion. Taking logs and integrating the stochastic differential of this price process over a time interval $[0; t+\Delta]$ yields

$$\ln P_{t+\Delta}^{dk} = \ln P_0^{dk} + \left[\mu^{dk} - \frac{1}{2}\left(\sigma^{dk}\right)^2\right](t+\Delta) + \sigma^{dk}\sqrt{t+\Delta}\,\mathbb{W}_{t+\Delta}^{dk}$$

where μ^{dk} and σ^{dk} correspond to the parameters in equation $(9a)$ for the domestic stock and to the parameters in equation $(9b)$ for the investment in the foreign bank-account process and the foreign stock. The log return for holding the asset over the interval $[t; t+\Delta]$ is

$$\ln R_{t+\Delta}^{dk} = \ln\left(\frac{P_{t+\Delta}^{dk}}{P_t^{dk}}\right) = \left[\mu^{dk} - \frac{1}{2}\left(\sigma^{dk}\right)^2\right]\Delta + \sigma^{dk}\sqrt{\Delta}\varepsilon_{t+\Delta}^{dk}$$

where $W_{t+\Delta}^{dk} - W_t^{dk} = \varepsilon_{t+\Delta}^{dk} \sim N(0,1)$. If the length of the holding period $[t, t+\Delta]$ is a day, we have a model for daily returns. Now we can take the model to data. With observations of daily log returns, the unconditional expectation and unconditional variance of the log return for the holding period are

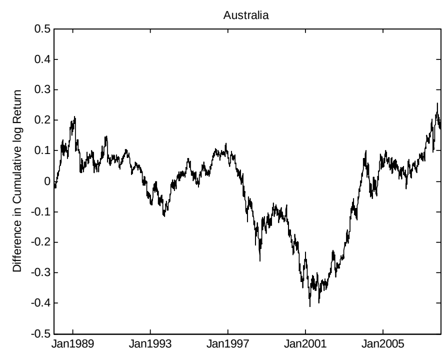
$$E\left[\ln R_t^{dk}\right] = \left[\mu^{dk} - \frac{1}{2}\left(\sigma^{dk}\right)^2\right]\Delta$$

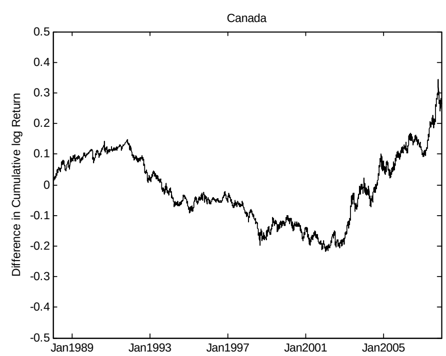
$$Var\left[\ln R_t^{dk}\right] = \left(\sigma^{dk}\right)^2\Delta$$

Therefore, I estimate the drift μ^{dk} and volatility σ^{dk} of the price process by the sample moments from the log return. From here, I can set up the market price of risk equations, derive the market prices of risk, and recover the SDF implied by these assets. Figure (1) presents the difference in the cumulative log return for the foreign stock market denominated in USD and the cumulative log return for the foreign stock market denominated in the foreign currency. For all six foreign countries, both cumulative log returns move closely. However, there are long periods for which the log returns separate. The difference is given by the foreign exchange which is stochastic.

Table (1) presents summary statistics. The left panel, in the first three columns, shows the parameter estimates for the excess return in USD for investing in the foreign bank-account process which is risky for an American investor. The second panel, in the last three columns, shows the parameters for the excess return in USD for investing in the foreign stock market. For each investment, I report the annualized estimate for the excess return, the volatility, and the correlation of the asset with the U.S. stock market. The estimates in the first column from the left panel are evidence of an exchange risk premium. If there is no foreign-exchange risk premium, these estimates should be zero. However, for Australia, Canada, and Japan, the estimates are significantly different from zero. The second column presents the estimates for the annualized volatility parameter; the volatility for most countries is around 1%, except for Canada, where it is lower and close to 0.6% The third column presents the correlation of the investment in the foreign bank-account process with the U.S. stock market. These correlations are low and generally negative, except for Australia and Canada, where the correlation is positive but still low. For investments in foreign stocks, the excess return in USD ranges from -5.63% in Japan to 5.3% in Switzerland. These excess returns are statistically significant. The volatilities for investments in the foreign stock are higher than the volatilities for investments in the foreign bank-account process, ranging from 1.54% to 2.25%. The last column presents the correlation of the log return for investing in the foreign stock and the log return for investing in the U.S. stock market. All correlations are positive and range from 5% for Japan to 63% for Canada.

Tables (2) and (3) present the main results. Table (2) presents the estimates for the annualized volatilities of the domestic and foreign SDFs and their correlation. The volatilities are similar in magnitude for the domestic and foreign economy, ranging between 30% and 40%. The higher difference in the volatilities of the SDF is for Switzerland; the volatility of the SDF


Australia
Difference in Cumulative log Return
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
Jan1989
Jan1993
Jan1997
Jan2001
Jan2005


Canada
Difference in Cumulative log Return
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
Jan1989
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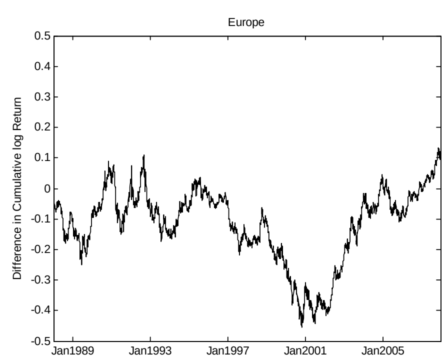
Europe
Difference in Cumulative log Return
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
Jan1989
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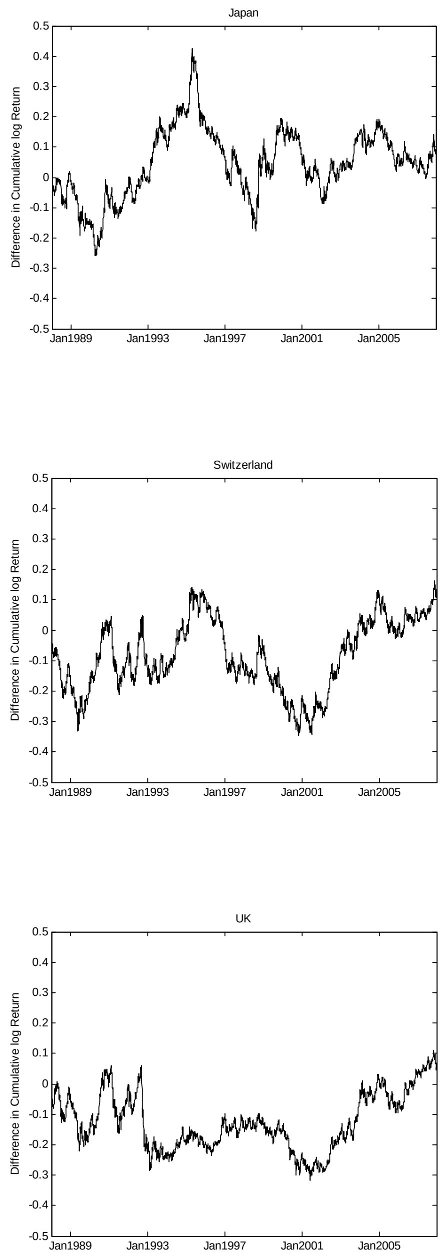

Figure 1: Difference between the cumulative log return in the foreign stock market in U.S. Dollars and the cumulative log return in the foreing stock market in the foreign currency. Sample is daily from January 1987 to December 2007

Table 1: Summary of statistics

	Foreign Bank Account			Foreign Stock		
	$\mu - r^r$	σ	ρ	$\mu - r^d$	σ	ρ
Australia	2.36	1.02	0.06	4.32	1.71	0.09
Canada	1.95	0.61	0.12	4.43	1.54	0.63
Europe	0.67	1.06	−0.11	4.97	1.94	0.33
Japan	−2.05	1.11	−0.09	−5.63	2.25	0.05
Switzerland	−1.01	1.14	−0.20	5.30	1.77	0.20
UK	1.53	0.95	0.09	2.48	1.62	0.31

Table 1: This table presents, in the left panel, the annualized excess return, volatility, and correlation with the U.S. stock return for investments in the foreign bank-account process. In the right panel, it presents the annualized excess return, volatility, and correlation with the U.S. stock return for investments in the foreign stock market. Samples are daily from January 1987 to December 2007.

in Swiss Francs is 33% and the corresponding volatility for the stochastic discount factor in USD is 42%. The third column presents the ratio of the volatilities, domestic to foreign. The ratios range from 0.85 for the U.K. to 1.35 for Switzerland. The fourth column presents the correlation between the SDFs. The correlations are always positive and above 50%, suggesting a strong correlation of the SDFs. Correlations range from 55% for Japan to as high as 97% for the UK.

Table (3) presents the correlation of the domestic stochastic discount factor with the domestic stock, the foreign exchange, and the foreign stock. The second column presents the correlation between the domestic SDF and the exchange rate. There is a wide variation in the correlations between the domestic SDF and the exchange rates. If exchange rate risk is relevant to price assets then we should expect that the domestic SDF and the the exchange rate to be correlated. Correlations range from −46% for Japan to 78% for Canada. While the absolute values of correlations between the exchange rate and the domestic assets is very low, the absolute values of the correlations between exchange-rate risk and the domestic SDF are generally high. Thus, even though exchange rates are not highly correlated with domestic stocks, exchange-rate risk is relevant to determine how assets are priced.

The rest of table (3) presents the correlation of the domestic stock with the domestic SDF and the foreign stock with the domestic SDF. The first

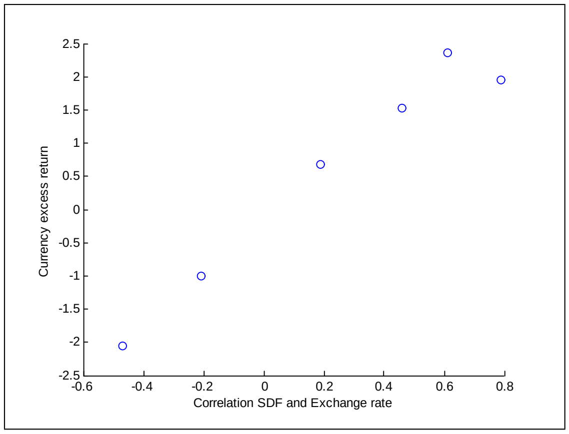

Figure 2: This figure presents, on the y-axis, the excess return for investing in the foreign bank account process and, on the x-axis, the correlation between the domestic SDF and the exchange rate risk given by equation (23).

column shows the correlation with the domestic stock. These correlations are all positive and consistently above 68%. The third column shows the correlation between the foreign stock and the domestic SDF, ranging from -46% for Japan to 91% for the U.K. The results presented in this table indicate that, in a globalized financial market, foreign factors are as important as domestic factors.

Finally, we study the relationship between the excess return for investing in the foreign bank-account process and the correlation between the exchange-rate risk and the domestic SDF. Figure (2) presents the excess return for investing in the foreign bank-account process from table (1) and the correlation between the domestic SDF and the exchange-rate risk from table (3). There is clearly a positive relationship between these two parameters. For currencies for which there is a risk premium (either positive or negative), the exchange-rate risk is strongly correlated with the domestic SDF.

Within this framework, the return on carry trades emerges as the ideal candidate to capture currency risk. Carry trades take long positions on currencies with high interest rates, which are known to have positive excess return, and short positions with currencies with low interest rates, which are known to have negative excess returns. Therefore, a carry trade is a portfolio of currencies in which exchange-rate risk is highly correlated with

Table 2: Stochastic Discount Factors

	Foreign	Domestic	Ratio	ρ
Australia	0.41	0.37	0.91	0.61
Canada	0.33	0.41	1.24	0.66
Europe	0.33	0.33	0.98	0.83
Japan	0.35	0.39	1.11	0.55
Switzerland	0.31	0.42	1.35	0.81
UK	0.40	0.34	0.85	0.97

Table 2: This table presents the annualized volatility of the domestic and foreign stochastic discount factors. The third column presents the ratio of volatilities defined as domestic over foreign. The last column presents the correlation between the stochastic discount factors.

Table 3: Correlation with SDF

	US Stock	Foreign Exchange	Foreign Stock
Australia	0.76	0.61	0.40
Canada	0.70	0.78	0.46
Europe	0.86	0.18	0.70
Japan	0.78	−0.46	−0.46
Switzerland	0.68	−0.20	0.91
UK	0.83	0.46	0.19

Table 3: This table presents the correlation between the domestic stochastic discount factor and the domestic stock, the foreign exchange, and the foreign stock.

the SDF. The traditional example of a carry trade is to borrow from Japan and to lend in Australia. In figure (2), Japan corresponds to the point in the lower left of the figure and Australia corresponds to the highest point in the right-hand side of the figure. In the appendix, I extend the model to include many countries and show how the results from the theoretical model extend to carry trades.

In a companion paper, I estimate a factor model for U.S. stock returns with the return on the carry trade as a factor. I provide strong empirical evidence for the relevance of this factor to price U.S. stock returns. The results from this model indicate that carry trades capture currency risk, and this risk reflects the differences in the SDFs for the U.S. and foreign countries.

5 Conclusions

This paper presents a continuous-time model of asset return processes in a global economy, based on the technique of change of numéraire. If all arbitrage opportunities have been eliminated, markets are complete and all assets are expressed in a common currency, then there is a unique stochastic discount factor process that prices all assets. However the representation of the stochastic discount factor depends on the currency used to express assets.

If there is a stochastic exchange rate process, then investments in foreign assets expressed in the domestic currency are exposed to currency risk. In this paper, I derive explicit formulas that express the impact of currency risk on excess returns. The model yields a continuous time analog of the uncovered interest rate parity condition, where there is an exchange rate premium with a clear economic interpretation; it reflects the relative volatility and the correlation between the stochastic discount factors. The premium compensates investor for all investments that are exposed to exchange-rate risk and I demonstrate that this risk can be represented by the return on the carry trade.

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A Appendix

A.1 Pricing equations

Here, I derive the market price of risk equation for the domestic economy in equations (12a) and (12b). What we need to show is that any asset denominated in the domestic currency $\Lambda_t^d P_t^{dk}$ is a martingale; that is, the drift of this process is equal to zero. First, consider a domestic asset $k \leq m$. Applying Ito's product rule to equations (11) and (9a) gives,

$$d\left(\Lambda_t^d P_t^{dk}\right) = \Lambda_t^d P_t^{dk} \left[\left(\mu_t^{dn} - r_t^d - \sum_{j=1}^{D} \sigma_t^{dnj} \theta^{dj} \right) dt + \sum_{j=1}^{D} \left(\sigma_t^{dnj} - \theta_t^{dj} \right) dW_t^j \right]$$

Therefore, $\left(\Lambda_t^d P_t^{dk}\right)$ is a martingale for a domestic asset if and only if

$$\mu_t^{dn} - r_t^d - \sum_{j=1}^{D} \sigma_t^{dnj} \theta^{dj}$$

Now consider a foreign asset expressed in domestic currency $k > m$. Applying Ito's product rule to equations (11) and (9b) gives,

$$d\left(\Lambda_t^d P_t^{dk}\right) = \Lambda_t^d P_t^{dk} \left[\begin{array}{c} \left(\mu_t^{fn} + \mu_t^Q + \sigma_t^{fn} \sigma_t^Q \rho_t^{fnQ} - r_t^d - \sum_{j=1}^{D} \left(\sigma_t^{fnj} + \sigma_t^{Qj} \right) \theta_t^{dj} \right) dt \\ + \sum_{j=1}^{D} \left(\sigma_t^{dnj} - \theta_t^{dj} \right) dW_t^j \end{array} \right]$$

Therefore, $\left(\Lambda_t^d P_t^{dk}\right)$ is a martingale for a foreign asset denominated in the domestic currency if and only if

$$\mu_t^{fn} + \mu_t^Q + \sigma_t^{fn} \sigma_t^Q \rho_t^{fnQ} - r_t^d = \sum_{j=1}^{D} \left(\sigma_t^{fnj} + \sigma_t^{Qj} \right) \theta_t^{dj}$$

.

A.2 Exchange Rate and SDF

Proof of proposition 1: Let Z_t^d be the Radon-Nikodym derivative that transforms the true probability measure $\mathbb{P}$ into the domestic risk-neutral probability measure $\mathbb{P}^d$, and Z_t^f the Radon-Nikodym derivative that transforms the true probability measure into the foreign risk-neutral probability measure $\mathbb{P}^f$, that is

$$P_0^{dk} = E\left[Z_t^d \frac{P_t^{dk}}{B_t^d}\right]$$

$$P_0^{fk} = E\left[Z_t^f \frac{P_t^{fk}}{B_t^f}\right]$$

Then the SDF for the foreign economy is $\Lambda_t^d = Z_t^d/B_t^d$, and the SDF for the foreign economy is $\Lambda_t^d = Z_t^d/B_t^d$. Consequently, what we need to prove is

$$\frac{Z_t^d}{B_t^d} = \frac{Z_t^d}{B_t^d}\frac{Q_t}{Q_0}$$

.

Because markets are complete and all arbitrage opportunities have been eliminated, Z_t^d and Z_t^f are the unique to the SDFs

$$dZ_t^d = Z_t^d\left[\sum_{j=1}^{D} \theta_t^{dj} dW_t^j\right],\ Z_0^d = 1 \tag{24}$$

$$dZ_t^f = Z_t^f\left[\sum_{j=1}^{D} \theta_t^{fj} dW_t^j\right],\ Z_0^f = 1 \tag{25}$$

Let P_t^{dk} be the price process for any asset denominated in the domestic currency and B^d the price of the domestic bank-account process. Then

$$P_0^{dk} = B_0^d E^d\left[\frac{P_t^{dk}}{B_t^d}\right] \tag{26}$$

where E^d is the expectation under the domestic risk-neutral probability measure. If we change the numéraire to the foreign bank-account process expressed in domestic currency $Q_t B_t^f$, then

$$P_0^{dk} = Q_0 B_0^f E^d\left[\frac{P_t^{dk}}{Q_t B_t^f} Z_t^{df}\right] \tag{27}$$

Since P_t^{dk}/Q_t is the price of the asset expressed in the foreign currency and B_t^f is the foreign bank-account process, Z_t^{df} is the Radon-Nikodym derivative that transforms the domestic risk-neutral measure into the foreign risk-neutral measure

$$Z_t^{df} = \frac{d\mathbb{P}^f}{d\mathbb{P}^d}$$

Then we have from (26) and (27)

$$B_0^d E^d \left[\frac{P_t^d}{B_t^d} \right] = Q_0 B_0^f E^d \left[\frac{P_t^d}{Q_t B_t^f} Z_t^{df} \right] \tag{28}$$

and the Radon-Nikodym derivative Z_t^{df} is

$$Z_t^{df} = \frac{B_t^f}{B_t^d} \frac{Q_t}{Q_0} \tag{29}$$

.By applying Ito's lemma to this expression then, under the true probability measure

$$dZ_t^{df} = Z_t^{df} \left[\left(r_t^f + \mu_t^Q - r_t^d \right) dt + \sum_{j=1}^{D} \sigma_t^{Qj} dW_t^j \right]$$

. Equation (28) under the true probability measure is

$$B_0^d E \left[Z_t^d \frac{P_t^d}{B_t^d} \right] = Q_0 B_0^f E \left[\frac{P_t^d}{Q_t B_t^f} Z_t^d Z_t^{df} \right]$$

.Therefore, since markets are complete, it must be that $Z_t^f = Z_t^d Z_t^{df}$, which from (29) implies

$$\frac{Z_t^f}{B_t^f} = \frac{Z_t^d}{B_t^d} \frac{Q_t}{Q_0}$$

Finally, by Ito's product rule

$$dZ_t^d Z_t^{df} = Z_t^d Z_t^{df} \left[\left(r_t^f + \mu_t^Q - r_t^d + \sum_{j=1}^{D} \theta_t^{dj} \sigma_t^{Qj} \right) dt + \sum_{j=1}^{D} \left(\sigma_t^{Qj} + \theta_t^{dj} \right) dW_t^j \right]$$

matching terms with (25) gives

$$\sigma_t^{Qj} = \theta_t^{fj} - \theta_t^{dj}$$

$$\mu_t^Q = r_t^d - r_t^f + \sum_{j=1}^{D} \theta_t^{dj} \left(\theta_t^{dj} - \theta_t^{fj} \right)$$

A.3 Foreign Exchange Risk Premium.

Here, I show that we can express the foreign-exchange risk premium for the domestic economy as in equation (20) . We begin from the expression for the foreign exchange risk premium in equation (16), derived in section 2.4:

$$\begin{aligned}\beta_t^d &= \sum_{j=1}^{D} \theta_t^{dj}\left[\theta_t^{dj} - \theta_t^{fj}\right] \\ &= \sum_{j=1}^{D}\left(\theta_t^{dj}\right)^2 - \sum_{j=1}^{D}\theta_t^{dj}\theta_t^{fj}\end{aligned}$$

The first term on the right-hand side corresponds to the square of the volatility of the domestic SDF in equation (18) and the second term is the co-variation of the domestic and foreign SDF. We can express the co-variation in terms of the correlation by

$$\rho_t^{f,d} = \frac{\sum_{j=1}^{D}\theta_t^{dj}\theta_t^{fj}}{\theta_t^d\theta_t^f}$$

,so we can write,

$$\beta_t^d = \left(\theta_t^d\right)^2 - \theta_t^d\theta_t^f\rho_t^{f,d}$$

.Finally, by multiplying and dividing the first term on the right-hand side by the volatility of the foreign stochastic discount factor θ_t^f, we arrive at (20)

$$\beta_t^d = \theta_t^d\theta_t^f\left(\frac{\theta_t^d}{\theta_t^f} - \rho_t^{f,d}\right)$$

A.4 Carry Trades

In this section, I modify the model from the previous section by assuming that there are I foreign countries. Each foreign country has a risk-free asset and a collection of risky assets driven by the $D-$dimensional Brownian motion W. Markets are complete, and all arbitrage opportunities have been eliminated. Let us further assume that the nominal exchange-rate process between the domestic country and the foreign country $i = [1, 2, ..., I]$ is also driven by the $D-$dimensional Brownian motion W and is given by

$$dQ_t^{f_i} = Q_t^{f_i}\left[\mu_t^{f_i}dt + \sigma_t^{f_i}d\mathbb{W}_t^{f_i}\right]$$

where $Q_t^{f_i}$ is the nominal exchange rate between the domestic country and the foreign country i. The nominal exchange rate is defined as the number of domestic currency units required to purchase one unit of country i's foreign currency. In the absence of arbitrage opportunities, the nominal exchange rate between two foreign countries is given by $Q^{ih} = Q^{f_i}/Q^{f_h}$, where Q^{ih} is the nominal exchange rate between foreign country i and foreign country h, defined as the number of currency units of country i required to purchase one unit of currency of country h. The stochastic process for the exchange rate Q^{ih} is, by Ito's product rule,

$$dQ_t^{ih} = Q_t^{ih}\left[\mu^{ih}dt + \sigma_t^{f_i}d\mathbb{W}_t^{f_i} - \sigma_t^{f_h}d\mathbb{W}_t^{f_h}\right]$$

where

$$\mu^{ih} = \mu_t^{f_i} - \mu_t^{f_h} + \left(\sigma^{f_h}\right)^2 - \sigma_t^{f_i}\sigma_t^{f_h}\rho^{ih}$$

ρ^{ih} is the correlation between the nominal exchange-rate processes $Q_t^{f_i}$ and $Q_t^{f_h}$. With the exchange rate processes, we can denominate all assets in the global economy in a single currency and derive the numéraire-dependent SDF for each country as in the model presented above. For any pair of countries, the corresponding stochastic discount factors yield the conditions for the parameters of the nominal exchange rate between that pair of countries.

To illustrate the carry trade, assume that $r^{f_i} < r^d < r^{f_h}$; that is, the risk-free interest rate in country i is lower than the domestic risk-free interest rate, which is lower than the risk-free interest rate in country h. A carry-trade strategy for a domestic investor consists of borrowing in country i to lend in country h. The value of the carry-trade portfolio at time t is $CT_t = Q_t^{f_h}B_t^{f_h} - Q_t^{f_i}B_t^{f_i}$, with the initial condition $CT_0 = 0$. The differential $d\left(CT_t\right)$ for the carry-trade value at time t depends on two components: the gain on the foreign bank-account process in the country with the high interest rate $Q_t^{f_h}B_t^{f_h}$ and the gain on the foreign bank-account process in the country with the low interest $Q_t^{f_i}B_t^{f_i}$: That is,

$$dCT_t = Q_t^{f_h}B_t^{f_h}\left[\left(\mu_t^{f_h} + r_t^{f_h}\right)dt + \sigma_t^{f_h}d\mathbb{W}_t^{f_h}\right] - Q_t^{f_i}B_t^{f_i}\left[\left(\mu_t^{f_i} + r_t^{f_i}\right)dt + \sigma_t^{f_i}d\mathbb{W}_t^{f_i}\right]$$

.We can write this equation more intuitively as

$$dCT_t = \left\{\begin{array}{c} CT_t r_t^d dt + \left(Q_t^{f_h}B_t^{f_h}\beta_t^{f_k} - Q_t^{f_i}B_t^{f_i}\beta_t^{f_i}\right)dt \\ +Q_t^{f_h}B_t^{f_h}\sigma_t^{f_h}d\mathbb{W}_t^{f_h} - Q_t^{f_i}B_t^{f_i}\sigma_t^{f_i}d\mathbb{W}_t^{f_i} \end{array}\right\} \tag{30}$$

.There are four terms appearing in equation (30) that can be interpreted in the following way: The first term, $CT_t r_t^d dt$, can be interpreted as the risk-free return on the value of the portfolio. The second term, $\left(Q_t^{f_h} B_t^{f_h} \beta_t^{f_k} - Q_t^{f_i} B_t^{f_i} \beta_t^{f_i}\right) dt$, can be interpreted as the premium for investing in the carry-trade strategy; the premium is equal to the value-weighted premiums of the currencies in the portfolio. The third term, $Q_t^{f_h} B_t^{f_h} \sigma_t^{f_h} d\mathbb{W}_t^{f_h}$, is the volatility of the portfolio proportional to the value of investment in the foreign bank-account process with the high interest rate. The last term, $Q_t^{f_i} B_t^{f_i} \sigma_t^{f_i} d\mathbb{W}_t^{f_i}$, is the volatility of the portfolio proportional to the value of the investment in the foreign bank-account process with the lowest interest rate.

As in the previous section, to determine how risky the strategy is, we look at the quadratic co-variation of the asset with the domestic SDF, which is given by

$$d\left[CT, \Lambda^d\right] = -\Lambda^d \left(Q_t^{f_h} B_t^{f_h} \beta_t^{f_k} - Q_t^{f_h} B_t^{f_h} \beta_t^{f_i}\right) dt$$

As for any other asset, there is a relation between the risk premium of the asset and the co-variation with the domestic SDF. Risky portfolios have positive risk premiums while hedge portfolios have negative premiums. The higher the premium, the stronger the co-variation with the domestic SDF, and the riskier the asset.