The difference indifference makes in strategy-proof allocation of objects*

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August 27, 2009

Abstract

We study the problem of allocating objects among people. We consider cases where each object is initially owned by someone, no object is initially owned by anyone, and combinations of the two. The problems we look at are those where each person has a need for exactly one object and initially owns at most one object (also known as "house allocation with existing tenants"). We split with most of the existing literature on this topic by dropping the assumption that people can always strictly rank the objects. We show that, without this assumption, problems in which either some or all of the objects are not initially owned are equivalent to problems where each object is initially owned by someone. Thus, it suffices to study problems of the latter type.

We ask if there are efficient rules that provide incentives for each person not only to participate (rather than stay home with what he owns), but also to state his preferences honestly. Our main contribution is to show that the answer is positive.

The intuitive "top trading cycles" algorithm provides such a rule for environments where people are never indifferent (Ma 1994). Our solution is a generalization of this algorithm that allows for indifference without compromising on efficiency and incentives.

JEL classification: C71, C78, D71, D78

Keywords: strategy-proofness, indivisible goods, indifference, housing market, house allocation, kidney exchange

*Acknowledgments to be added.

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1 Introduction

Consider a setting where each person in a group has a need for a single object (such as a seminar slot, an on-campus apartment, or an organ for transplant) and may or may not be endowed with such an object. Further, suppose that there are no divisible goods, such as money. Even when every person is endowed with an object, the initial distribution is not necessarily efficient.

When people are never indifferent between objects, there are *strategy-proof, Pareto-efficient,* and *individually rational* rules (Abdulkadiroğlu and Sönmez 1999). In fact, a group of such rules is characterized by these three axioms with the help of consistency and neutrality axioms (Sönmez and Ünver 2008).

At one extreme of this class of problems are those where nobody is endowed with an object and there is only a social endowment (Hylland and Zeckhauser 1979). Again, when people are never indifferent, the class of rules satisfying *group strategy-proofness* and *Pareto-efficiency* have been characterized (Pycia and Ünver 2009).

At the other extreme are problems where everybody is endowed with an object but there is no social endowment (Shapley and Scarf 1974). For these problems, when people are never indifferent between objects, there are rules with desirable efficiency and incentive properties. The *core* contains a unique allocation which is also the unique *competitive allocation* (Roth and Postlewaite 1977). The rule that maps each problem with its unique core allocation is not only *strategy-proof* (Roth 1982) but also *group strategy-proof* (Bird 1984). Further, it is the only *strategy-proof, Pareto-efficient,* and *individually rational* rule (Ma 1994, Sönmez 1999). It is also *non-bossy* and *anonymous* (Miyagawa 2002).

We argue that there are many real-world situations where people's preferences do exhibit indifference. For instance, if preferences are based on coarse descriptions (perhaps from a housing brochure), there may be insufficient information to break ties. Alternatively, if preferences are based on checklists of criteria (like blood-type and genetic markers for organ transplant), distinct objects satisfying exactly the same criteria are equivalent. Appropriate design of rules should take these indifferences into account since breaking ties arbitrarily may lead to inefficiency.

We show that when we drop the assumption that people are never indifferent, all of the problems mentioned above can be thought of as ones where every person is endowed with an object. Thus, we study only such problems. Without strict preferences, many of the results mentioned above no longer hold. Though the *weak core* is not empty, the *core* is no longer guaranteed to exist (Shapley and Scarf 1974).[1] The set of *competitive allocations* no longer coincides with the *core*

[1] Quint and Wako (2004) provide necessary and sufficient conditions on preference profiles for the core to be non-empty.

(Wako 1991). *Group strategy-proofness* and *Pareto-efficiency* are incompatible (Ehlers 2002).

We show that there may not even be an *efficient competitive allocation*. We provide a direct proof that *strategy-proofness*, *Pareto-efficiency*, and *individual rationality* are not compatible with *non-bossiness*.[2] Further, we show that, even when we drop *individual rationality*, they are not compatible with *anonymity*.

Our main contribution is to show that *strategy-proof, Pareto-efficient, and individual rational rules do exist.* We do so by defining adaptations of the "top trading cycles" algorithm, and then showing that the associated rules satisfy these properties.

The remainder of the paper is organized as follows. We present the model in Section 2. We describe some desiderata of allocations and rules in Section 3 and define our rules, along with some others, in Section 4. In Section 5 we present our results. We show how the more general problems involving social endowments can be encoded as problems with only private endowments in Section 6. We conclude in Section 7.

2 The Model

Let $\boldsymbol{O}$ be a set of distinct objects. Let $\boldsymbol{N}$ be a set of people. There are exactly as many objects as people: $|O| = |N|$. An **endowment** is a bijection, $\omega : N \to O$, that associates an object with each person. For each $i \in N$, i's component of the endowment is $\omega(i)$. Each person has a preference relation over O. Let the set of all preference relations be $\mathcal{R}$. A **preference profile** associates each individual with a preference relation in $\mathcal{R}$. Let $\mathcal{R}^N$ be the set of all preference profiles. Given a profile $R \in \mathcal{R}^N$, for each $i \in N$, i's preference relation is R_i. For each pair of alternatives, $a, b \in O$, if i finds a to be at least as good as b, we write $a\ R_i\ b$. If a is better than b, that is, $a\ R_i\ b$ but not $b\ R_i\ a$, we write $a\ P_i\ b$. Similarly, if i is indifferent between a and b, we write $a\ I_i\ b$. Let $\mathcal{P} \subset \mathcal{R}$ be the set of "strict" preference profiles. That is, $\mathcal{P} \equiv \{R_0 \in \mathcal{R} : \text{ for each } a, b \in O, a\ I_0\ b \Leftrightarrow a = b\}$.

We use the notation R_{-i} to denote the preference relations of everyone but i. For each group $S \subseteq N$, we denote the preferences of all the people in S by R_S, and those not in S by R_{-S}. We denote the set of all preferences for people in the group S by $\mathcal{R}^S$.

Let $\boldsymbol{A}$, the set of all bijections from N to O, be the set of all possible allocations. For each $\alpha \in A$, and each $i \in N$, let $\alpha(i)$ denote i's component of α. Similarly,

[2] Bogomolnaia, Deb and Ehlers (2005) show this by characterizing, for problems with no private endowment, classes of *strategy-proof* and *Pareto-efficient* rules satisfying two different forms of *non-bossiness* and some auxillary axioms.

for each $S \subseteq N$, let $\alpha(S)$ be the collective assignment to members of S under α. That is, $\alpha(S) = \bigcup_{i \in S}\{\alpha(i)\}$.

A **problem** consists of a preference profile and an endowment, $(R, \omega) \in \mathcal{R}^N \times A$. A **rule**, $\varphi : \mathcal{R}^N \times A \to A$, selects an allocation for each problem.

3 Properties of allocations and rules

In this section, we list some desiderata of allocations and rules. Let φ be a rule.

The first requirement is that a rule respects each individual's endowment. That is, the allocation selected by the rule should not assign, to any person, an object that he finds worse than his endowment.

For each $(R, \omega) \in \mathcal{R}^N \times A$ and $\alpha \in A$, we say that **α is individually rational at (R, ω)** if for each $i \in N, \alpha(i)\ R_i\ \omega(i)$. Let $\boldsymbol{IR(R, \omega)}$ be the set of all *individually rational allocations at* (R, ω).

Individual Rationality: For each $(R, \omega) \in \mathcal{R}^N \times A$, $\varphi(R, \omega) \in IR(R, \omega)$.

Before we state the next requirement, we define an efficiency relation between allocations. For each $\alpha, \beta \in A$ and $R \in \mathcal{R}^N$, **α Pareto dominates β at R** if at least one person is better off at α than at β and nobody is worse off. That is, for some $i \in N$, $\alpha(i)\ P_i\ \beta(i)$ and for each $i \in N, \alpha(i)\ R_i\ \beta(i)$.

For each $R \in \mathcal{R}^N$, let the set of allocations that are not *Pareto dominated* by any other allocation be $\boldsymbol{PE(R)}$.

Pareto-efficiency: For each $(R, \omega) \in \mathcal{R}^N \times A$, $\varphi(R, \omega) \in PE(R)$.

The next property is that misreporting one's preferences is never beneficial.

Strategy-proofness: For each $(R, \omega) \in \mathcal{R}^N \times A$, each $i \in N$, and each $R'_i \in \mathcal{R}$,

$$\varphi(\underbrace{R_i}_{\text{truth}}, R_{-i}, \omega)(i) \underbrace{R_i}_{\text{truth}} \varphi(\underbrace{R'_i}_{\text{lie}}, R_{-i}, \omega)(i).$$

The following is the requirement that nobody can affect what the rule assigns to others without affecting his own assignment.

Non-bossiness: For each $R \in \mathcal{R}^N$, each $\omega \in A$, each $i \in N$, and each $R'_i \in \mathcal{R}$,

$$\varphi(R, \omega)(i) = \varphi(R'_i, R_{-i}, \omega)(i) \Rightarrow \varphi(R, \omega) = \varphi(R'_i, R_{-i}, \omega).$$

The next desideratum is that the rule is a function of preferences and endowments, but not identities. Let $\pi : N \to N$ be a permutation of N. For each

$(R, \omega) \in \mathcal{R}^N \times A$, define the **permutation of $\boldsymbol{R}$ with respect to $\boldsymbol{\pi}$ and $\boldsymbol{\omega}$, $\boldsymbol{R^{\pi,\omega}}$** $\in \mathcal{R}^N$, such that for each $i, j, k \in N$,

$$\omega(j)\ R_i\ \omega(k) \Leftrightarrow \omega(\pi(j))\ R^{\pi,\omega}_{\pi(i)}\ \omega(\pi(k)).$$

Anonymity: For each $i, j \in N$, each $R \in \mathcal{R}^N$, $\omega \in A$, and each $\pi : N \to N$,[3]

$$\varphi(R^{\pi,\omega}, \omega) = \varphi(R, \omega).$$

The final requirement is that no group of people would rather re-allocate their endowments among themselves than participate in the application of the rule. This can be expressed in two ways. First, for each $\alpha \in A$, $R \in \mathcal{R}^N$, $\omega \in A$, and $S \subseteq N$, we say that **$\boldsymbol{\alpha}$ is blocked by $\boldsymbol{S}$** if members of S can re-allocate their endowments in a way that makes each of them better off than at α. That is, there is $\beta \in A$ such that $\beta(S) = \omega(S)$, for each $i \in S, \beta(i)\ P_i\ \alpha(i)$. Second, we say that **$\boldsymbol{\alpha}$ is weakly blocked by $\boldsymbol{S}$** if members of S can re-allocate their endowments in a way that makes at least one of them better off than at α, while none of the rest are made worse off than at α. That is, there is $\beta \in A$ such that $\beta(S) = \omega(S)$, for some $i \in S, \beta(i)\ P_i\ \alpha(i)$, and for each $i \in S, \beta(i)\ R_i\ \alpha(i)$.

The **weak core, $\boldsymbol{C^W(R, \omega)}$**, is the set of allocations that are not *blocked* by any coalition and the **core, $\boldsymbol{C(R, \omega)}$**, is the set of allocations that are not *weakly blocked* by any coalition.

4 Rules

Let $\prec$ be a linear ordering of N.

Sequential priority rules: Let a *tie-breaker* $\theta : \mathbb{P}(A) \setminus \{\varnothing\} \to A$ be such that for each $A' \subseteq A$, $\theta(A') \in A'$.[4] The **sequential priority rule with respect to $\prec$ and $\boldsymbol{\theta}$, $\boldsymbol{SP^{\prec,\theta}}$,** is defined as follows. Suppose $\prec$ is such that $1 \prec 2 \prec \cdots \prec n$.

For each $(R, \omega) \in \mathcal{R}^N \times A$, we define a sequence of subsets of A, $\{A_i^{R,\omega}\}_{i=0}^{i=n}$. Let $A_0^{R,\omega} = A$. For each $i = 1, \ldots n$,

$$A_i^{R,\omega} = \{\alpha \in A_{i-1}^{R,\omega} :\ \text{for each } \beta \in A_{i-1}^{R,\omega}, \alpha(i)\ R_i\ \beta(i)\}.$$

Finally, define $SP^{\prec}(R, \omega) = \theta(A_n^{R,\omega})$.[5] $\triangle$

Sequential priority rules are *strategy-proof* and *Pareto-efficient* (Svensson 1994). But they are not *individually rational*.

[3]This formulation of *anonymity* is taken from Miyagawa (2002).
[4]Given a set S, we denote the *power set of S* by $\mathbb{P}(S)$.
[5]This definition is taken from Svensson (1994).

Sequential priority selections from IR: Let $g : \mathbb{P}(A) \to A$ be a function that selects an allocation from each subset of A. The **sequential priority selection from IR with respect to $\prec$ and g, $SP\text{-}IR^{\prec,g}$,** is defined exactly as $SP^{\prec,g}$ except that $A_0^{R,\omega} = IR(R,\omega)$. △

Sequential priority selections from IR are not *strategy-proof* but are *Pareto-efficient* and, by definition, *individually rational.*

The notion of "most preferred" objects among a subset of O is critical for the definition of our next rule. For each $R \in \mathcal{R}^N$, $O' \subseteq O$, and $i \in N$, let **i's most preferred objects, under R_i, among O', $\tau(R_i, O')$** $\equiv \{a \in A :$ for each $b \in O', a \; R_i \; b\}$.

Gale's "top trading cycles" algorithm (Shapley and Scarf 1974) is applicable after breaking ties arbitrarily. The associated rules are *strategy-proof* and *individually rational* but not *Pareto-efficient.* The next class of rules that we define are based on an an adaptation of this algorithm.[6]

Top cycles rules: For each $R \in \mathcal{R}^N$ and $\omega \in A$, we define the allocation selected by **top cycles rule with priority $\prec$, $TC^{\prec}(R,\omega)$,** via the following algorithm.

At every step, for each person we check if he "stays" or "leaves" with what he holds, based on an *efficiency* condition. Then, every person who remains "points" at another person who holds one of his most preferred objects among those remaining. Objects are traded according to cycles of pointing people.

The goal of our algorithm is to enlarge, at each step, the "satisfied" people: those holding one of their most preferred objects among those remaining. However, this is to be done in a way that provides incentives for every person to report his true preferences. To provide such incentives, the algorithm favors people who have higher priority by connecting more people to them (via direct or indirect pointing) as compared to people with lower priority.

Note: The description of the algorithm is fairly involved. The reader may find it useful to concurrently refer to the extensive example that we have provided after the formal description of the algorithm.

Formally, for each step $t = 0, 1, 2, \ldots$, we define the **remaining objects, $O_t \subseteq O$**, the **remaining people, $N_t \subseteq N$**, and the next step's **holding vector, $h_{t+1} : N_t \to O_t$**. We also define, for each $i \in N_t$, the **person whom i points at, $p_t(i)$**. For notational convenience, for each $i, j \in N_t$, we use $\boldsymbol{i} \underset{\boldsymbol{t}}{\longrightarrow} \boldsymbol{j}$ to denote $p_t(i) = j$. If $p_t(p_t(i)) = j$, we write $\boldsymbol{i} \underset{\boldsymbol{t}}{\longrightarrow}\underset{\boldsymbol{t}}{\longrightarrow} \boldsymbol{j}$, and so on. Given $M \subseteq N_t$, if $p_t(i) \in M$, we write $\boldsymbol{i} \underset{\boldsymbol{t}}{\longrightarrow} \boldsymbol{M}$. If $p_t(p_t(i)) \in M$, we write $\boldsymbol{i} \underset{\boldsymbol{t}}{\longrightarrow}\underset{\boldsymbol{t}}{\longrightarrow} \boldsymbol{M}$, and so on.

[6] However, we have the departure phase at the beginning of each step rather than at the end. We have done this for expositional simplicity since it rules out people "pointing" at themselves.

For (O_t, N_t, h_t), let the **satisfied people, S_t** be the set of people who hold one of their most preferred objects among O_t. That is, $S_t \equiv \{i \in N_t : h_t(i) \in \tau(R_i, O_t)\}$. Let the **unsatisfied people, U_t** $\equiv N_t \setminus S_t$.

Let $O_0 \equiv O$, $N_0 \equiv N$, and $h_1 \equiv \omega$.

At step $t = 1, 2, \ldots$, we get (O_t, N_t, h_{t+1}, p_t) as follows.

Departure phase: We define $\{N_t^k\}_{k=0}^K$ and $\{G_t^k\}_{k=1}^K$ as follows. Let $N_t^0 = N_{t-1}$. For each $k = 1, 2, \ldots$, let $G_t^k \subseteq N_t^{k-1}$ be the *largest*[7] set such that for each $i \in G_t^k$,

$$\begin{array}{rl} \text{i)} & h_{t-1}(i) \in \tau(R_i, h_{t-1}(N_t^{k-1})) \text{ and} \\ \text{ii)} & \tau(R_i, h_{t-1}(N_t^{k-1})) \subseteq h_{t-1}(G_t^k). \end{array}$$

Let $N_t^k = N_t^{k-1} \setminus G_t^k$.

Finally, let K be such that $G_t^K = \varnothing$ and for each $k < K$, $G_t^k \neq \varnothing$.

Then, each $i \in \bigcup_{l \leq k} G_t^k$, departs with $h_t(i)$. That is, $TC^{\prec}(R, \omega)(i) \equiv h_t(i)$. Further,

$$\begin{array}{rcl} N_t & \equiv & N_t^K \text{ and} \\ O_t & \equiv & h_t(N_t).^8 \end{array}$$

Pointing phase: We determine p_t in stages, as follows: For each $i \in N_t$, let i's **candidate pointees, $C_{i,t}$**$\equiv \{j \in N_t : h_t(j) \in \tau(R_i, O_t)\}$.

Stage 1) If $t \neq 1$, we first consider $i \in N_t$ such that i's pointee in Step $t-1$ still remains and holds the same object as he did at Step $t-1$. Then, i points at the same person in Step t as well. That is, if $t \neq 1$, for each $i \in N_t$ such that $i \underset{t-1}{\longrightarrow} j \in N_t$, and $h_t(j) = h_{t-1}(j)$, we have $i \underset{t}{\longrightarrow} j$.

Stage 2) Now, we consider $i \in N_t$ that has only one candidate pointee. He points at his unique candidate pointee. That is, for each $i \in N_t$ such that $C_{i,t} = \{j\}$, we have $i \underset{t}{\longrightarrow} j$.

Stage 3) Next, we consider $i \in N_t$ with at least one unsatisfied candidate pointee. He points at the unsatisfied candidate pointee with highest priority.[9] That is,

$$p_t(i) \equiv \arg \underset{j \in C_{i,t} \setminus S_t}{\prec\text{-max}}\, j.^{10}$$

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[7]To obtain this largest set, start with the N_t^{k-1} and eliminate one member violating the condition at a time.

[8]If $K = 1$ then $N_t = N_{t-1}$.

[9]The order within a stage is unimportant because stages are performed sequentially, and if $p_t(i)$ is defined at Stage k, then $p_t(p_t(i))$ is defined at Stage $k' < k$. Further, $p_t(i)$ is independent of $p_t(j)$ if $p_t(j)$ is defined at Stage $k'' \geq k$.

Stage 4) Now, we consider $i \in N_t$ with only satisfied candidate pointees, at least one of whom has an unsatisfied pointee. He points at the satisfied candidate whose unsatisfied pointee has highest priority (breaking ties with respect to $\prec$). That is,

$$\begin{aligned} C^1_{i,t} &\equiv \{j \in C_{i,t} : j \underset{t}{\longrightarrow} U_t\} \subseteq S_t, \\ J_t(i) &\equiv \arg \prec\text{-}\max_{j \in C^1_{i,t}} p_t(j), \text{ and} \\ p_t(i) &\equiv \arg \prec\text{-}\max_{j \in J_t(i)} j. \end{aligned}$$

Stage 5) Next, we consider $i \in N_t$ whose candidate pointees are all satisfied and have satisfied pointees, at least one of whom has an unsatisfied pointee. He points at the candidate who points at the person who points at the unsatisfied person with highest priority (again, breaking ties with $\prec$). That is,

$$\begin{aligned} C^2_{i,t} &\equiv \{j \in C_{i,t} : j \underset{t}{\longrightarrow}\underset{t}{\longrightarrow} U_t\} \subseteq S_t, \\ J_t(i) &\equiv \arg \prec\text{-}\max_{j \in C^2_{i,t}} p_t(p_t(j)), \text{ and} \\ p_t(i) &\equiv \arg \prec\text{-}\max_{j \in J_t(i)} j. \end{aligned}$$

Stage ...) The process is repeated until for each $i \in N_t$, $p_t(i)$ is defined.

By definition of the departure phase, each $i \in N_t$ points, directly or indirectly, at an unsatisfied person. Thus, the pointing phase terminates in a finite number of stages.

Trading phase: There is at least one cycle $C \equiv \{i_1, i_2, \ldots, i_s\}$ such that $i_1 \underset{t}{\longrightarrow} i_2 \underset{t}{\longrightarrow} \ldots \underset{t}{\longrightarrow} i_s \underset{t}{\longrightarrow} i_1$. Further, each $i \in N_t$ is a member of at most one cycle. We get h_{t+1} by performing the trades prescribed by each cycle. That is, for each cycle, $\{i_1, i_2, \ldots, i_s\}$, and each $k = 1, \ldots s$, $h_{t+1}(i_{k-1}) = h_t(i_k)$. For each $i \in N_t$ who is not in a cycle, $h_{t+1}(i) = h_t(i)$.

The algorithm terminates at Step $\mathring{t}$ such that $N_{\mathring{t}} = \varnothing$. △

Since the algorithm terminates and provides a unique allocation for every problem, $TC^{\prec}$ is a well-defined rule. To see this, note that at each step, since N is finite and there is at least one cycle involving an unsatisfied person, either

[10] For each $f : X \to N$ and each $X' \subseteq X$, we define $\arg \prec\text{-}\max_{j \in X'} f(j) \equiv i \in X'$ such that for each $j \in X' \setminus \{i\}, f(i) \prec f(j)$.

1. At least one person departs with his holding, or
2. At least one person's holding is switched to an object that he ranks highest among those remaining. Therefore, for each t, $U_{t+1} \subseteq U_t$.

Therefore, the algorithm terminates in a finite number of steps.

Remark 1. (Evolving priority orders) An interpretation of the pointing phase is that the priority order is updated at every stage to demote satisfied people relative to unsatisfied people. We define the rule in this way to achieve *Pareto-efficiency*.

If the priority is unchanging, and the satisfied people all have higher priority than unsatisfied people, they may trade among themselves indefinitely. In such a case, due to the condition for departure, which is required to achieve *Pareto-efficiency*, the algorithm will not terminate. ◦

To help illustrate the *top cycles rule*, we provide an example.

Example 1. *Top cycles rule.*

Let $O = \{a, b, c, d, e, f, g, h, i, j\}$, and $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Consider $(R, \omega) \in \mathcal{R}^N \times A$ such that $\omega = (a, b, c, d, e, f, g, h, i, j)$ and $R \in \mathcal{R}^N$ as follows:

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}
a	a	f	$c\ d\ e\ f$	$d\ e\ g$	$d\ e$	d	$d\ h\ i$	$c\ i$	$a\ b\ j$
$\vdots$	b	$\vdots$	$\vdots$	$\vdots$	$\vdots$	g	$\vdots$	$\vdots$	$\vdots$
	$\vdots$					$\vdots$			

Let $\prec$ be such that $1 \prec 2 \prec 3 \prec 4 \prec 8 \prec 5 \prec 6 \prec 7 \prec 9 \prec 10$.

We start with $O_0 = O, N_0 = N$, and $h_1 = \omega$.

Step 1:

Departure phase: The sequence satisfying the departure condition is $G_1^1 = \{1\}$ and $G_1^2 = \{2, 10\}$. To see this, note that 1's most preferred object in O is the unique object a, his endowment. Given that 1 leaves with a, 2's most preferred object in $O \setminus \{a\}$ is the unique object b and 10's most preferred objects in $O \setminus \{a\}$ are b and j. Now, $TC^{\prec}(R, \omega)(1) = a$, $TC^{\prec}(R, \omega)(2) = b$, and $TC^{\prec}(R, \omega)(10) = j$. Further, $N_1 = \{3, 4, 5, 6, 7, 8, 9\}$, and $O_1 = \omega(N_1)$. From this, $S_1 = \{4, 5, 8, 9\}$.

Pointing phase: This is illustrated in Figure 1.

Stage 1) Not applicable to the first step.

Stage 2) Each person with a unique most preferred object in O_1 points at whomever holds that object. In this case, $3 \underset{1}{\longrightarrow} 6$ and $7 \underset{1}{\longrightarrow} 4$.

Stage 3) Each person such that one of their most preferred objects is held by an unsatisfied person points at an unsatisfied person. Such people are 4, 5, and 9. Since 5 and 9 have only one unsatisfied person to point at, they point accordingly. That is, $5 \underset{1}{\longrightarrow} 7$ and $9 \underset{1}{\longrightarrow} 3$. However, 4 is indifferent between the objects held by 3 and 6. In accordance with $\prec$, $4 \underset{1}{\longrightarrow} 3$.

Stage 4) Each person whose most preferred objects are held by satisfied people with unsatisfied pointees point. 6 and 8 are such people. Since 4 and 5 hold 6's most preferred objects, we consider who their pointees are. Since $4 \underset{1}{\longrightarrow} 3, 5 \underset{1}{\longrightarrow} 7$, and $3 \prec 7$, we have $6 \underset{1}{\longrightarrow} 4$ rather than $6 \underset{1}{\longrightarrow} 5$. 8's candidate-pontees are 9 and 4 who both point at 3. Since $4 \prec 9$, $8 \underset{1}{\longrightarrow} 4$.

Trading phase: We observe that there is only one cycle and it involves 3, 4, and 6. Thus, $h_2 = (-, -, f, c, e, d, g, h, i, -)$.

Step 2:

Departure phase: The sequence satisfying the departure condition is $G_2^1 = \{3\}$. From this, $TC^{\prec}(R, \omega)(3) = f$, $N_2 = \{4, 5, 6, 7, 9\}$, $O_2 = \{c, d, e, g, h, i\}$, and $S_2 = \{4, 5, 6, 8, 9\}$.

Pointing phase: This is illustrated in Figure 2.

Stage 1) Since $5 \underset{1}{\longrightarrow} 7 \in N_2$ and $h_2(7) = h_1(7)$, we have $5 \underset{2}{\longrightarrow} 7$.

Stage 2) Since 7's unique most preferred object is d, $7 \underset{2}{\longrightarrow} 6$.

Stage 3) No person, other than 5, most prefers g (7's holding) among O_2.

Stage 4) 4 and 6 point at 5 whose pointee is unsatisfied: $4 \underset{2}{\longrightarrow} 5$ and $6 \underset{2}{\longrightarrow} 5$.

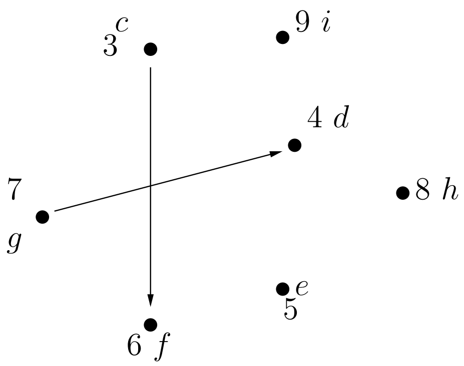
Stage 5) The pointee of the pointees of 8 and 9 is 5. Thus, $8 \underset{2}{\longrightarrow} 6$ and $9 \underset{2}{\longrightarrow} 4$.

Trading Phase: At the end of this Step, there is one cycle and it involves 5, 6, and 7. In the *trading phase*, we get $h_3 = (-, -, -, c, g, e, d, h, i, -)$.

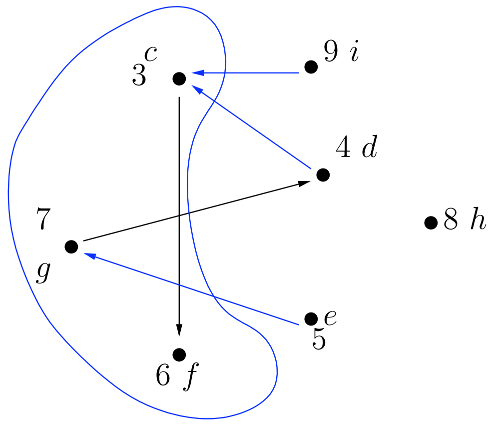
Step 3:

Departure phase: We end after the departure phase of Step 3 since $G_3^1 = \{4, 5, 6, 7, 8, 9\}$ and $N_3 = \varnothing$.

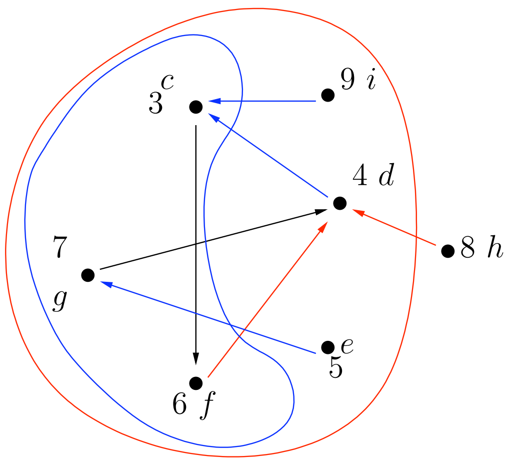
Thus, $TC^{\prec}(R, \omega) = (a, b, f, c, g, e, d, h, i, j)$. •



(a)



(b)



(c)

Figure 1: Pointing phase of Step 1: (a) Since 3 and 7 have unique most preferred objects, they point at whoever holds those objects. (b) Next, we consider 4, 9 and 5: those who have a most preferred object that is held by an unsatisfied person in the bubble. (c) Finally, we consider 6 and 8: those who have a most preferred object that is held by a member of the bigger bubble: people who can point at an unsatisfied person.

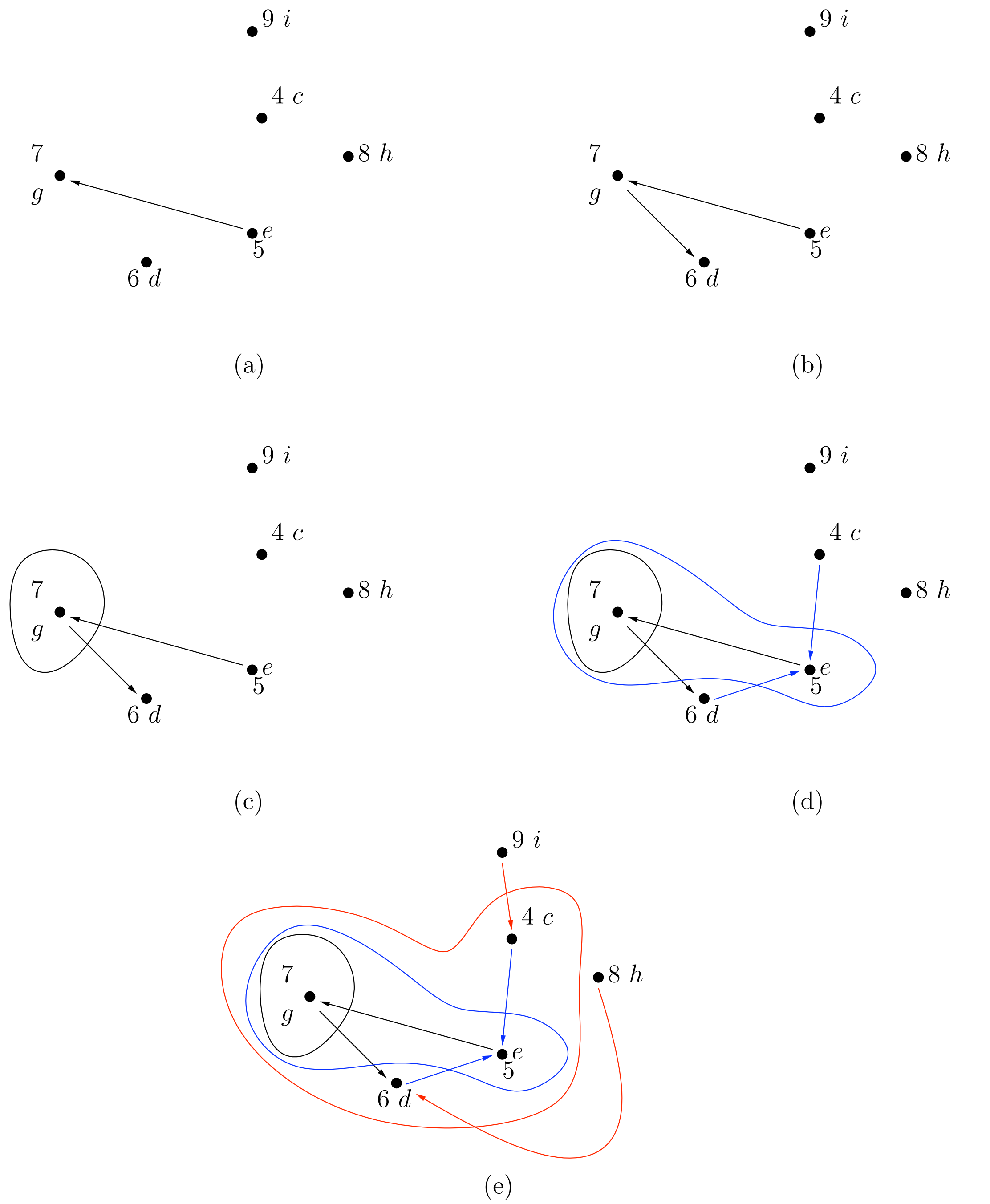


Figure 2: Pointing phase of Step 2: (a) Since 5 pointed at 7 in step 1 and 7 has not traded, 5 points at 7 in Step 2 as well. (b) 7 is the only person with a unique most preferred object. (c) We now consider people who can point at the only unsatisfied person, 7. However, there is no such person. (d) Next, we consider 4 and 6 who point into the bubble containing 7 and 5. (e) Finally, 8 and 9 point into the biggest bubble.

In the next section, we show that $TC^{\prec}$ is *strategy-proof*, *Pareto-efficient*, and *individually rational*. We also show that $TC^{\prec}$ always picks an allocation from the *weak core*.

When the input preference profile does not involve any indifference, the priority order $\prec$ plays no role in the definition of $TC^{\prec}$ since for each $i \in N$ and each t, $\tau(R_i, O_t)$ is a singleton and $p_t(i)$ is defined in the first two stages of the *pointing phase*. Thus, for each $(P, \omega) \in \mathcal{P}^N \times A$ and each pair of priority orders $\prec$ and $\prec'$, $TC^{\prec}(P, \omega) = TC^{\prec'}(P, \omega)$.

Remark 2. It is natural to ask whether, for each $(R, \omega) \in \mathcal{R}^N \times A$, there is a corresponding problem $(P', \omega) \in \mathcal{P}^N \times A$ such that,

1. For each $i \in N$ and each pair $x, y \in O$, if $x \; P'_i \; y$, then $x \; R_i \; y$, and
2. $TC^{\prec}(R, \omega) = TC^{\prec}(P', \omega)$.

However, this is not the case. Consider the following example. Let $N = \{1, 2, 3\}$, $\omega \equiv (a, b, c)$, and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3
$b\ c$	a	a
a	c	b
	b	c

For each $\prec$ such that $2 \prec 3$, $TC^{\prec}(R, \omega) = (c, a, b)$. There are exactly two profiles $P^1, P^2 \in \mathcal{P}^N$ that meet the above condition (1.):

P^1_1	P^1_2	P^1_3
b	a	a
c	c	b
a	b	c

P^2_1	P^2_2	P^2_3
c	a	a
b	c	b
a	b	c

But $TC^{\prec}(P^1, \omega) = (b, a, c)$ and $TC^{\prec}(P^2, \omega) = (c, b, a)$, neither of which coincides with $TC^{\prec}(R, \omega)$. [11] ○

[11] The reason for this is that regardless of how we break ties, the result of the *top-trading cycles* algorithm is a "competitive allocation" (Shapley and Scarf 1974). In fact every *competitive allocation* can be found in this way. However, as evidenced by the above example, for some profiles of preferences, no *competitive allocation* is *Pareto-efficient* and $TC^{\prec}(R, \omega)$ need not be a competitive allocation.

5 Results

We first show that *strategy-proofness* and *Pareto-efficiency* are incompatible with *anonymity*. We also show that the additional requirement of *individual rationality* leads to an incompatibility with *non-bossiness*. We then show that these incompatibilities are tight by proving that *top cycles* rules satisfy all three of our central axioms. The proofs of these results are in the appendix.

Proposition 1. *If $N > 2$, no rule is strategy-proof, Pareto-efficient and anonymous.*

Proposition 2. *If $N > 2$, no rule is strategy-proof, Pareto-efficient, individually rational, and non-bossy.*[12]

Next, we show that both Propositions 2 and 1 are tight. When we drop *non-bossiness* or *anonymity* from the list of requirements, the incompatibility does not persist, as evidenced by *top cycles* rules.

By definition *top cycles* rules are not anonymous. To see that they are *bossy*, consider the following example.

Example 2. Bossiness of top cycles rules: Let $O = \{a, b, c\}, N = \{1, 2, 3\}, \omega = (a, b, c)$, and $1 \prec 2 \prec 3$. Let $R, R' \in \mathcal{R}^N$ be such that,

R_1	R_2	R_3
$a\ b\ \textcircled{c}$	$\textcircled{a}$	a
	b	$\textcircled{b}$
	c	c

R'_1	R_2	R_3
$\textcircled{c}$	a	$\textcircled{a}$
$a\ b$	$\textcircled{b}$	b
	c	c

.

Then, $TC^{\prec}$ selects the circled allocations above, showing that it is *bossy*. •

Proposition 3. *For each priority order $\prec$, $TC^{\prec}$ is Pareto-efficient and individually rational. That is, for each $(R, \omega) \in \mathcal{R}^N \times A$ and each $\prec$, $TC^{\prec}(R, \omega) \in PE(R) \cap IR(R, \omega)$.*

Proof: By definition of $TC^{\prec}$, it is *individually rational.*

We show that it is *Pareto-efficient*. Consider the sequence $\{G_1^k\}_{k=1}^K$ of people who leave at the first step. Each member of G_1^1 leaves with one of his most preferred objects and can be made no better off. Each member of G_1^2 leaves with one of his most preferred objects after members of G_1^1 have left and can be made no better off without hurting at least one member of G_1^1. Continuing, each member of G_1^{K-1} leaves with one of his most preferred objects after members of $G_1^1 \cup G_1^2 \cup \cdots \cup G_1^{K-2}$

[12]This is a corollary of Theorem 2 in (Bogomolnaia et al. 2005). We provide a direct proof in the appendix.

have left and can be made no better off without hurting at least one person who has left.

A similar argument applies to the subsequent steps. Those leaving in later steps can be made no better off without hurting those who have left in prior steps. Thus, $TC^{\prec}$ is *Pareto-efficient*. ■

Proposition 4. *For each priority order $\prec$, $TC^{\prec}$ selects an allocation from the weak core. That is, for each $(R,\omega) \in \mathcal{R}^N \times A$ and each $\prec$, $TC^{\prec}(R,\omega) \in C^W(R,\omega)$.*

Proof: Suppose not. Then, there are $(R,\omega) \in \mathcal{R}^N \times A$ and $S \subseteq N$ *such that* S *blocks* $\alpha \equiv TC^{\prec}(R,\omega)$. That is, there is $\beta \in A$ such that $\beta(S) = \omega(S)$ and for each $i \in S$, $\beta(i)\ P_i\ \alpha(i)$.

For each t and each $i \in S$, if $\beta(i) \in O_t$, then $i \underset{t}{\not\longrightarrow} j$ such that $\beta(i)\ P_i\ h_t(j)$.

Let $\hat{t}$ be the first step at which there is $i \in S$ such that i is part of a trading cycle at the end of step $\hat{t}$. Then $h_{\hat{t}+1}(i)\ I_i\ \alpha(i)$. So $\beta(i)\ P_i\ h_{\hat{t}+1}(i)$. This implies that $i \underset{\hat{t}}{\longrightarrow} j$ such that $\beta(i)\ P_i\ h_{\hat{t}}(j) = h_{\hat{t}+1}(i)$. Thus, $\beta(i) \notin O_{\hat{t}}$. However, since $\beta(S) = \omega(S)$, there is $k \in S$ such that $\beta(i) = \omega(k)$ and since $\omega(k) \notin O_{\hat{t}}$, k is part of a trading cycle at some $\tilde{t} < \hat{t}$. This contradicts the definition of $\hat{t}$. ■

In order to show that for each $\prec$, $TC^{\prec}$ is *strategy-proof*, we make a preliminary remark and show two key lemmas.

For each problem $(R,\omega) \in \mathcal{R}^N \times A$, the "state" of the algorithm at Step t is summarized by the tuple (O_t, N_t, h_{t+1}, p_t). Our remark and lemmas pertain to how these tuples change in response to changes in the input problem. Though they seem monotonous, these lemmas provide useful insight into the dynamics of the algorithm.

Remark 3. (Persistence) *If i points at j at Step t, then he points at j as long as j holds the same object. That is, if $i \underset{t}{\longrightarrow} j$, then for every $t' > t$ such that $h_{t'}(j) = h_{t'-1}(j) = h_{t'-2}(j) = \ldots = h_t(j)$, $i \underset{t'}{\longrightarrow} j$.*

Before we proceed to our first lemma, we introduce some additional notation. Let $(R,\omega) \in \mathcal{R}^N \times A$ and $i \in N$. At the Step t of the algorithm, let the set of people **connected to $\boldsymbol{i}$, $\boldsymbol{CONN(i,R,t)}$**, be those, including i, connected to i via p_t. That is,

$$CONN(i,R,t) = \left\{ j \in N_t : \begin{array}{l} j \equiv i, \text{ or} \\ j \underset{t}{\longrightarrow} i, \text{ or} \\ j \underset{t}{\longrightarrow} \underset{t}{\longrightarrow} i, \text{ or} \\ \cdots \end{array} \right\}.$$

Fix $\omega \in A$ and priority order $\prec$. Let $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$. Let $R' = (R'_i, R_{-i})$. For each $\hat{t} = 0, 1, \ldots$, let $h_{\hat{t}}$ be the *holding vector* at Step $\hat{t}$ of the algorithm for the problem (R, ω). Similarly define $h'_{\hat{t}}$ for the problem (R', ω). We also define, $O_{\hat{t}}, O'_{\hat{t}}$, $N_{\hat{t}}, N'_{\hat{t}}, p_{\hat{t}}, p'_{\hat{t}}, S_{\hat{t}}, S'_{\hat{t}}$, $U_{\hat{t}}$, and $U'_{\hat{t}}$. Finally, for each $i, j \in N$, we indicate $p_{\hat{t}}(i) = j$ by $i \xrightarrow[\hat{t}]{R} j$ and $p'_{\hat{t}}(i) = j$ by $i \xrightarrow[\hat{t}]{R'} j$. We also use $i \not\xrightarrow[\hat{t}]{R} j$ to indicate $p_{\hat{t}}(i) \neq j$.

Let t be the step at which i either leaves or makes his first trade under R. Define t' similarly with respect to R'. Let $\bar{t}$ be the first step at which i is satisfied for exactly one of the two problems if such a number exists, and ∞ otherwise. That is, $i \in S_{\bar{t}}$ and $i \in U'_{\bar{t}}$, $i \in U_{\bar{t}}$ and $i \in S'_{\bar{t}}$, or $\bar{t} = \infty$.[13]

Let $\underline{t} \equiv \min\{t, t', \bar{t}\}$.

Our first lemma states that up to Step $\underline{t}$, there is no difference in the *state* of the algorithm, regardless of whether i reports R_i or R'_i.

Lemma 1. ($\underline{t}$ equality) *At $\underline{t}$, for both R and R', the objects and people remaining, as well as the holding vector and previous step's pointing vector, except for i's component, are the same. That is,*

$$\begin{matrix} O_{\underline{t}} & N_{\underline{t}} & h_{\underline{t}} \\ \shortparallel & \shortparallel & \shortparallel \\ O'_{\underline{t}} & N'_{\underline{t}} & h'_{\underline{t}} \end{matrix} \quad \text{and for each } j, k \in N_{\underline{t}} \text{ such that } j \neq i, \quad \begin{matrix} j \xrightarrow[\underline{t}-1]{R} k \\ \Updownarrow \\ j \xrightarrow[\underline{t}-1]{R'} k. \end{matrix}$$

Proof:

Step 1: Since $i \in U_1$ and $i \in U'_1$, and for each $j \in N \setminus \{i\}$, $R_j = R'_j$ and $h_1(j) = h'_1(j) = \omega(j)$, we have $S_0 = S'_0$. Thus, $O_1 = O'_1$ and $N_1 = N'_1$. Therefore, for each $j \in N_1 \setminus \{i\}$, $p_1(j) = p'_1(j)$.

If $1 < \underline{t}$, i does not trade at Step 1 under either R or R'. Therefore, the cycles formed under p_1 and p'_1 are the same and do not involve i. Then, for each $j \in N_1$, $h_2(j) = h'_2(j)$.

Step 2: Since $i \in U_1$ and $i \in U'_1$, and for each $j \in N_1 \setminus \{i\}, R_j|_{O_1} = R'_j|_{O_1}$ and $h_2(j) = h'_2(j)$, we have $S_1 = S'_1$. Thus, $O_2 = O'_2$ and $N_2 = N'_2$.

As an **induction hypothesis**, suppose that for some $\ddot{t} < \underline{t}$, $O_{\ddot{t}} = O'_{\ddot{t}}$, $N_{\ddot{t}} = N'_{\ddot{t}}$, $h_{\ddot{t}} = h'_{\ddot{t}}$, and for each $j \in N_{\ddot{t}} \setminus \{i\}$, $p_{\ddot{t}-1}(j) = p'_{\ddot{t}-1}(j)$.

Step $\ddot{t} + 1$: We show that for $\ddot{t} < \underline{t}$, $O_{\ddot{t}+1} = O'_{\ddot{t}+1}$, $N_{\ddot{t}+1} = N'_{\ddot{t}+1}$, $h_{\ddot{t}+1} = h'_{\ddot{t}+1}$, and for each $j \in N_{\ddot{t}} \setminus \{i\}$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$.

[13] The last case, $\bar{t} = \infty$, only occurs if $t = t'$.

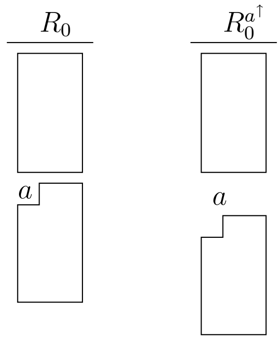


Figure 3: *Local push-up* of a preference relation: Given $R_0 \in \mathcal{R}$ and $a \in O$, the *local push-up of* R_0 *at* a, $R_0^{a\uparrow}$ is as shown above.

Since $\ddot{t} < \underline{t}$, $i \in U_{\ddot{t}}$ and $i \in U'_{\ddot{t}}$. In addition, by our *induction hypothesis*, $O_{\ddot{t}} = O'_{\ddot{t}}$, $N_{\ddot{t}} = N'_{\ddot{t}}$, $h_{\ddot{t}} = h'_{\ddot{t}}$ and $R_j|_{O_{\ddot{t}}} = R'_j|_{O_{\ddot{t}}}$. Thus, for each $j \in N_{\ddot{t}} \setminus \{i\}$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$.

Since $\ddot{t} < \underline{t}$, i does not trade under R or R' at $\ddot{t}$. Therefore, the cycles formed by $p_{\ddot{t}}$ and $p'_{\ddot{t}}$ are the same and do not involve i. Thus, for each $j \in N_{\ddot{t}}$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$. Also, $O_{\ddot{t}+1} = O'_{\ddot{t}+1}$ and $N_{\ddot{t}+1} = N'_{\ddot{t}+1}$. ♣

For each $R_0 \in \mathcal{R}$, and each $a \in O$, let the **indifference class of $\boldsymbol{a}$ at $\boldsymbol{R_0}$, $\boldsymbol{I(a, R_0)}$**, be

$$I(a, R_0) = \{b \in O \mid b \; I_0 \; a\}.$$

Given $R_0 \in \mathcal{R}$, and $a \in O$, define the **local push-up of $\boldsymbol{R_0}$ at $\boldsymbol{a}$, $\boldsymbol{R_0^{a\uparrow}}$** $\in \mathcal{R}$ be the relation that it differs from R_0 *only* in that it ranks a above all objects in $I(a, R_0)$, as shown in Figure 3. That is,

$$R_0|_{O\setminus\{a\}} = R_0^{a\uparrow}|_{O\setminus\{a\}} \text{ and for each } b \in O \setminus \{a\}, \begin{array}{l} b \; P_0 \; a \Rightarrow b \; P_0^{a\uparrow} \; a \text{ and} \\ a \; R_0 \; b \Rightarrow a \; P_0^{a\uparrow} \; b. \end{array}$$

To prove that $TC^{\prec}$ is *strategy-proof* we will have to consider all possible preference relations that a person can misreport. However, we can split all the available misreports into two categories. The first category includes only preference relations under which the person is not indifferent between the object that he is assigned and any other object. The second category consists of all the remaining preference relations. The following lemma implies that for each preference relation in the second category, we can find a preference relation in the first category such that the person is assigned the same object regardless of which of the two preference relations he reports. We use this to prove that $TC^{\prec}$ is *strategy-proof* since it means that we only need to rule out successful misreports from the first category.

Lemma 2. (Invariance) *If the preference relation of a person changes to a local push-up of his original preference at his assignment, then his assignment is unchanged. That is, if* $\alpha = TC^{\prec}(R,\omega)$, $R'_i = R_i^{\alpha(i)^{\uparrow}}$, *and* $\alpha' = TC^{\prec}(R',\omega)$, *then* $\alpha(i) = \alpha'(i)$.

Proof: By the $\underline{t}$ *equality* lemma, $O_{\underline{t}} = O'_{\underline{t}}$, $N_{\underline{t}} = N'_{\underline{t}}$, $h_{\underline{t}} = h'_{\underline{t}}$, and for each $j \in N_{\underline{t}} \setminus \{i\}$, $p_{\underline{t}-1}(j) = p'_{\underline{t}-1}(j)$. Since $O_{\underline{t}} = O'_{\underline{t}}$, by the definition of R_i and R'_i, $\tau(R'_i, O_{\underline{t}}) = \{\alpha(i)\} \subseteq \tau(R_i, O_{\underline{t}})$.
The $\underline{t}$ *equality* lemma also implies that $CONN(i,R,\underline{t}-1) = CONN(i,R',\underline{t}-1)$.

The rest of the proof proceeds as follows. First we show that at $\min\{t,t'\}$, any person connected to i under R is connected to i under R'. Then, we show that i's component of the allocation chosen under R is the same as his component of the allocation chosen under allocation under R': $\alpha'(i) = \alpha(i)$.

Claim 1. ***(Pre-trade inclusion)***[14] *For each* $\ddot{t} = \underline{t}, ..., \min\{t,t'\}$,[15]

(i) The objects and people remaining at $\ddot{t}$ *under* R *are a subset of those remaining under* R'. *Further, those remaining under* R' *but not under* R *are connected to* i. *That is,*

$$O_{\ddot{t}} \subseteq O'_{\ddot{t}}, \qquad N_{\ddot{t}} \subseteq N'_{\ddot{t}}$$
$$O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i,R',\ddot{t}-1)), \text{ and } \quad N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i,R',\ddot{t}-1).$$

(ii) Every person who is satisfied at $\ddot{t}$ *under* R' *is satisfied under* R. *Every person who is not satisfied under* R' *but is satisfied under* R *is connected to* i *under* R'. *That is,* $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ *and* $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i,R',\ddot{t}-1)$.

(iii) Every person not connected to i *at* $\ddot{t}$ *under* R' *points at the same person under* R *as under* R'. *That is, for each* $j \in N'_{\ddot{t}} \setminus CONN(i,R',\ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$.

(iv) Every person not connected to i *at* $\ddot{t}$ *under* R' *holds the same object under* R *as under* R'. *That is, for each* $j \in N'_{\ddot{t}} \setminus CONN(i,R',\ddot{t})$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$.

(v) The set of people connected to i *under* R *is a subset of the people connected to* i *under* R'. *That is,* $CONN(i,R,\ddot{t}) \subseteq CONN(i,R',\ddot{t})$.

Proof: Suppose $\underline{t} \neq \min\{t,t'\}$. Then $\underline{t} = \bar{t}$. Since $\tau(R'_i, O_{\bar{t}}) = \{\alpha(i)\}$, $\underline{t} < t'$, and by definition of $\bar{t}$, $i \in U'_{\bar{t}}$ and $i \in S_{\bar{t}}$.

Let $\ddot{t} = \bar{t}$. Statements (i) and (ii), for $\bar{t}$, are implied by the $\underline{t}$ *equality* lemma. Further, $S_{\bar{t}} = S'_{\bar{t}} \cup \{i\}$.

[14] As illustrated in Figure 4.

[15] If $\underline{t} = \min\{t,t'\}$ statements (i) - (v) are implied by the $\underline{t}$ *equality* lemma.

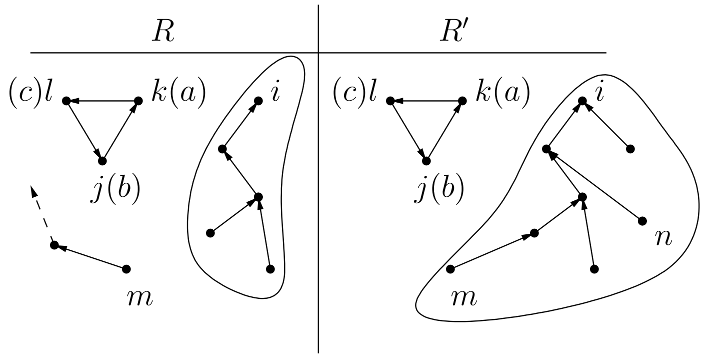
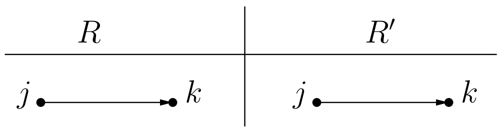


Figure 4: Pre-trade inclusion.

We now prove statement (iii), for $\bar{t}$, by following the progression of the pointing phase.[16] By the $\underline{t}$ *equality* lemma, each $j \in N_{\bar{t}} \setminus \{i\}$ pointed at the same person under R as he did under R' at step $\bar{t}-1$.

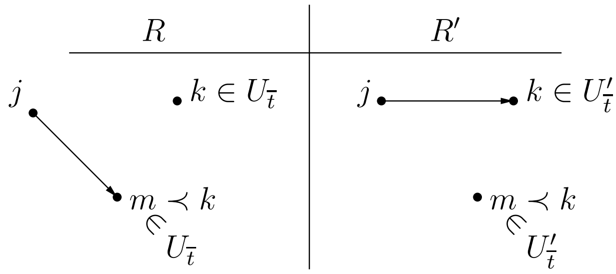
Stage 1) At the beginning of the pointing phase we consider people who were pointing at someone who remains in $N_{\bar{t}}$ and holds the same object. In particular, we consider $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$ such that $j \xrightarrow[\bar{t}-1]{R'} k \in N'_{\bar{t}}$ and $h'_{\bar{t}}(k) = h'_{\bar{t}-1}(k)$. Then, $j \xrightarrow[\bar{t}]{R'} k$. By the $\underline{t}$ *equality* lemma, $j \xrightarrow[\bar{t}-1]{R} k$ and $h_{\bar{t}}(k) = h_{\bar{t}-1}(k) = h'_{\bar{t}-1}(k)$. Thus $j \xrightarrow[\bar{t}]{R} k$.



Stage 2) Now we consider people who have a unique most preferred object. They point at the same person under R as under R'.

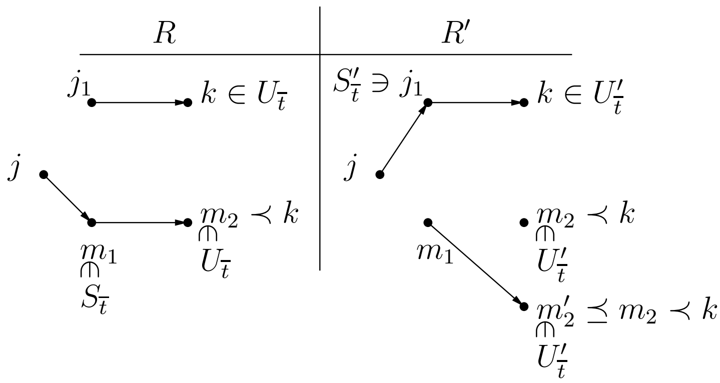
Stage 3) Next, we consider the people who point at unsatisfied people under R'. In particular, $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$ such that $j \xrightarrow[\bar{t}]{R'} k \in U'_{\bar{t}}$. Since $j \notin CONN(i, R', \bar{t})$, $k \notin CONN(i, R', \bar{t})$. Since $k \in U'_{\bar{t}}$ and $S_{\bar{t}} = S'_{\bar{t}} \cup \{i\}$, $k \in U_{\bar{t}}$. Further, $h_{\bar{t}}(k) = h'_{\bar{t}}(k) = \omega(k)$. Suppose $j \xrightarrow[\bar{t}]{R} m \neq k$. Then, $m \in U_{\bar{t}} \subseteq U'_{\bar{t}}$ and so $h_{\bar{t}}(m) = h'_{\bar{t}}(m) = \omega(m)$ and $m \prec k$. This contradicts $j \xrightarrow[\bar{t}]{R'} k$.

[16] We provide a graphical illustration of the argument following each stage of the pointing phase.



Stage 4) We now consider the people who point at satisfied people with unsatisfied pointees, under R'. In particular, we consider $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$ such that $j \xrightarrow[\bar{t}]{R'} j_1 \in S'_{\bar{t}} \xrightarrow[\bar{t}]{R'} k \in U'_{\bar{t}}$. Then, by (ii), $j_1 \in S_{\bar{t}}$.

By the preceding arguments, $j_1 \xrightarrow[\bar{t}]{R} k$ and $k \in U_{\bar{t}}$. Suppose $j \xrightarrow[\bar{t}]{R} m_1 \neq j_1$. If $m_1 \in U_{\bar{t}}$, then $h_{\bar{t}}(m_1) = h'_{\bar{t}}(m_1) = \omega(m_1)$ and $m_1 \in U'_{\bar{t}}$. But this contradicts $j \xrightarrow[\bar{t}]{R'} S'_{\bar{t}}$. So $m_1 \in S_{\bar{t}}$ and $m_1 \xrightarrow[\bar{t}]{R} m_2$ such that $m_2 \in U_{\bar{t}}$ and $m_2 \prec k$. Then, $m_2 \in U'_{\bar{t}}$ and thus $m_1 \xrightarrow[\bar{t}]{R'} m'_2 \in U'_{\bar{t}}$ such that $m'_2 \preceq m_2 \prec k$. [17] By the $\underline{t}$ *equality* lemma, $h_{\bar{t}}(m_1) = h'_{\bar{t}}(m_1)$. This contradicts $j \xrightarrow[\bar{t}]{R'} j_1$.

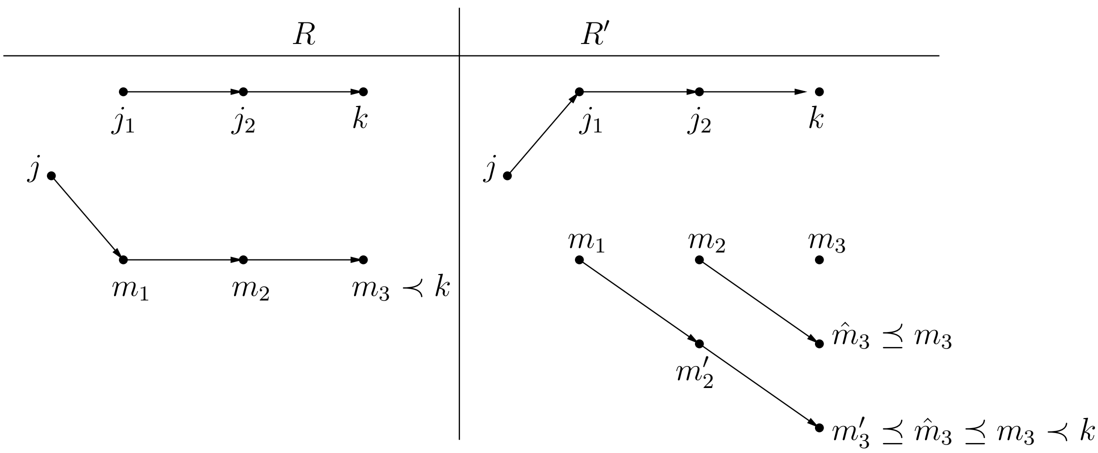


Stage 5) Now we consider the people who point at satisfied people with satisfied pointees whose pointees are unsatisfied, under R'. Particularly, consider $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$ be such that $j \xrightarrow[\bar{t}]{R'} j_1 \in S'_{\bar{t}} \xrightarrow[\bar{t}]{R'} j_2 \in S'_{\bar{t}} \xrightarrow[\bar{t}]{R'} k \in U'_{\bar{t}}$. Then, $j_1, j_2 \in S_{\bar{t}}$.

By the preceding arguments, $j_1 \xrightarrow[\bar{t}]{R} j_2 \xrightarrow[\bar{t}]{R} k \in U_{\bar{t}}$. Suppose $j \xrightarrow[\bar{t}]{R} m_1 \neq j_1$. If $m_1 \in U_{\bar{t}}$, then $h_{\bar{t}}(m_1) = h'_{\bar{t}}(m_1) = \omega(m_1)$ and $m_1 \in U'_{\bar{t}}$. But this contra-

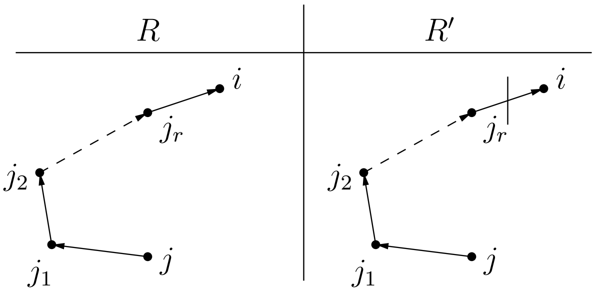
[17] We use the notation $i \preceq j$ to indicate $i \prec j$ or $i = j$.

dicts $j \xrightarrow[\bar{t}]{R'} S'_{\bar{t}}$. So $m_1 \in S_{\bar{t}}$. By the $\underline{t}$ *equality* lemma, $h_{\bar{t}}(m_1) = h'_{\bar{t}}(m_1)$. Let $m_1 \xrightarrow[\bar{t}]{R} m_2$. If $m_2 \in U_{\bar{t}}$, then $h_{\bar{t}}(m_2) = h'_{\bar{t}}(m_2) = \omega(m_2)$ and $m_2 \in U'_{\bar{t}}$. So $m_1 \xrightarrow[\bar{t}]{R'} U'_{\bar{t}}$. But this contradicts $j \xrightarrow[\bar{t}]{R'} S'_{\bar{t}} \xrightarrow[\bar{t}]{R'} S'_{\bar{t}}$. So $m_2 \in S_{\bar{t}}$. By the $\underline{t}$ *equality* lemma, $h_{\bar{t}}(m_2) = h'_{\bar{t}}(m_2)$. Since $k \in U_{\bar{t}}$, $m_2 \xrightarrow[\bar{t}]{R} m_3 \in U_{\bar{t}}$ and $m_3 \prec k$. Then, $m_3 \in U'_{\bar{t}}$ and so $m_2 \xrightarrow[\bar{t}]{R'} \hat{m}_3 \in U'_{\bar{t}}$ such that $\hat{m}_3 \preceq m_3 \prec k$. If $m_1 \xrightarrow[\bar{t}]{R'} m_2$, this contradicts $j \xrightarrow[\bar{t}]{R'} j_1$. Then $m_1 \xrightarrow[\bar{t}]{R'} m'_2 \neq m_2$ and $m'_2 \xrightarrow[\bar{t}]{R'} m'_3$. Note that $m'_2 \in S'_{\bar{t}}$, otherwise this contradicts $j \xrightarrow[\bar{t}]{R'} S'_{\bar{t}} \xrightarrow[\bar{t}]{R'} S'_{\bar{t}}$. In addition, since $m_1 \not\xrightarrow[\bar{t}]{R'} m_2$ and $m_1 \xrightarrow[\bar{t}]{R'} m'_2$, we have $m'_3 \in U'_{\bar{t}}$ and $m'_3 \prec \hat{m}_3$. Then, $m_3 \prec k$, which contradicts $j \xrightarrow[\bar{t}]{R'} j_1$.



Stage ...) Repeating this argument for the rest of the pointing phase, we show (iii).

We show that (v) $CONN(i, R, \bar{t}) \subseteq CONN(i, R', \bar{t})$ is a consequence of (iii). To see this, suppose $j \in CONN(i, R, \bar{t}) \setminus CONN(i, R', \bar{t})$. Then, there is a sequence $\{j_1, j_2, ..., j_r, i\} \subset N_{\bar{t}}$, such that $j \xrightarrow[\bar{t}]{R} j_1 \xrightarrow[\bar{t}]{R} j_2 \xrightarrow[\bar{t}]{R} ... \xrightarrow[\bar{t}]{R} j_r \xrightarrow[\bar{t}]{R} i$. Since $j \notin CONN(i, R', \bar{t})$, then by (iii), $j \xrightarrow[\bar{t}]{R'} j_1$. Then, $j_1 \notin CONN(i, R', \bar{t})$. Again, by (iii), $j_1 \xrightarrow[\bar{t}]{R'} j_2$ and $j_2 \notin CONN(i, R', \bar{t})$. Repeating the argument r times, $j_r \notin CONN(i, R', \bar{t})$. By (iii), $j_r \xrightarrow[\bar{t}]{R'} i$, and this contradicts $j \notin CONN(i, R', \bar{t})$.



Finally, we prove (iv) for Step $\bar{t}$. We show that for each $j \in N'_{\bar{t}} \backslash CONN(i, R', \bar{t})$, $h_{\bar{t}+1}(j) = h'_{\bar{t}+1}(j)$. Note that since at $\bar{t} < t'$, i does not trade. Then, no *trading cycle* under R' involves i. So, no *trading cycle* involves any member of $CONN(i, R', \bar{t})$. That is, for each *trading cycle* $C' \subset N'_{\bar{t}}$, $CONN(i, R', \bar{t}) \cap C' = \varnothing$. By (iii) and since $N_{\bar{t}} = N'_{\bar{t}}$, we have $C' \subset N_{\bar{t}}$ is also a *trading cycle* under R. Therefore, for each $j \in N'_{\bar{t}} \backslash CONN(i, R', \bar{t})$, $h'_{\bar{t}+1}(j) = h_{\bar{t}+1}(j)$. Moreover, for each $j \in CONN(i, R', \bar{t})$, $h'_{\bar{t}+1}(j) = h'_{\bar{t}}(j)$.

As an **induction hypothesis**, suppose that for some $\ddot{t} \in \{\underline{t}, ..., \min\{t, t'\} - 1\}$,

(i) $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$, $N_{\ddot{t}} \subseteq N'_{\ddot{t}}$
$O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i, R', \ddot{t} - 1))$, and $N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i, R', \ddot{t} - 1)$,

(ii) $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ and $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i, R', \ddot{t} - 1)$,

(iii) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$,

(iv) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$, and

(v) $CONN(i, R, \ddot{t}) \subseteq CONN(i, R', \ddot{t})$.

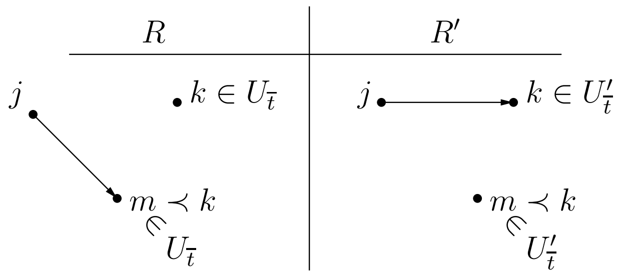
We prove that these statements are true of $\ddot{t}+1$. To prove (i) and (ii) for $\ddot{t}+1$, note that by (iv) and (v) of the *induction hypothesis*, if $C \in N_{\ddot{t}}$ is a *trading cycle* under R and is not a *trading cycle* under R', then $C \subseteq CONN(i, R', \ddot{t})$. Thus, at Step $\ddot{t} + 1$, we have statements (i) and (ii).

We now prove (iii), for $\ddot{t}+1$, by following the progression of the pointing phase just as in the case of $\bar{t}$.

Stage 1) We consider people whose pointee at $\ddot{t}$ remains at $\ddot{t} + 1$ and holds the same object under R as R'. In particular, we consider $j \in N'_{\ddot{t}+1} \backslash CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}]{R'} k \in N'_{\ddot{t}}$ and $h'_{\ddot{t}+1}(k) = h'_{\ddot{t}}(k)$. Then, $j \xrightarrow[\ddot{t}+1]{R'} k$. By the *induction hypothesis*, $j \xrightarrow[\ddot{t}]{R} k$ and $h_{\ddot{t}+1}(k) = h_{\ddot{t}}(k) = h'_{\ddot{t}}(k)$. Thus $j \xrightarrow[\ddot{t}+1]{R} k$.

Stage 2) Now we consider people who have a unique most preferred object. For each $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$, if $\tau(R_j, O'_{\ddot{t}+1}) = \{a\}$, then by the *induction hypothesis*, $h^{-1}_{\ddot{t}+1}(a) = h'^{-1}_{\ddot{t}+1}(a) \notin CONN(i, R', \ddot{t}+1)$. Thus, $a \in O_{\ddot{t}+1}$ and so $p_{\ddot{t}+1}(j) = p'_{\ddot{t}+1}(j)$.

Stage 3) Next, we consider the people with unsatisfied pointees under R'. In particular, $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}}$. Since $j \notin CONN(i, R', \ddot{t}+1)$, $k \notin CONN(i, R', \ddot{t}+1)$. Since $k \in U'_{\ddot{t}+1}$ and $S_{\ddot{t}+1} \setminus S'_{\ddot{t}+1} \subseteq CONN(i, R', \ddot{t}+1)$, $k \in U_{\ddot{t}+1}$. Further, $h_{\ddot{t}+1}(k) = h'_{\ddot{t}+1}(k) = \omega(k)$. Suppose $j \xrightarrow[\ddot{t}+1]{R} m \neq k$. Then, $m \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$ and so $h_{\ddot{t}+1}(m) = h'_{\ddot{t}+1}(m) = \omega(m)$ and $m \prec k$. This contradicts $j \xrightarrow[\ddot{t}+1]{R'} k$.



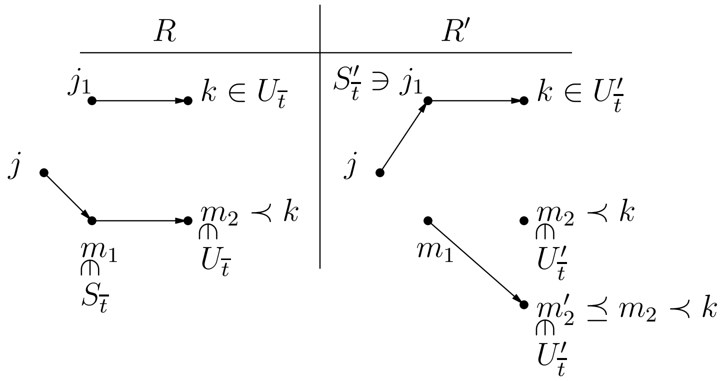
Stage 4) We now consider the people who point at satisfied people with unsatisfied pointees, under R'. In particular, we consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}+1}$. Then, by (ii), $j_1 \in S_{\ddot{t}+1}$.

By the preceding arguments, $j_1 \xrightarrow[\ddot{t}+1]{R} k$ and $k \in U_{\ddot{t}+1}$. Suppose $j \xrightarrow[\ddot{t}+1]{R} m_1 \neq j_1$. We consider the following two cases.

$h'_{\ddot{t}+1}(m_1) = h_{\ddot{t}+1}(m_1)$: If $m_1 \in U_{\ddot{t}+1}$, then $m_1 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_1) = h'_{\ddot{t}+1}(m_1) = \omega(m_1)$.

Then, $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_1 \in S_{\ddot{t}+1}$. Suppose $m_1 \xrightarrow[\ddot{t}+1]{R} m_2$. Since $j \xrightarrow[\ddot{t}+1]{R} m_1$ and $k \in U_{\ddot{t}+1}$, then $m_2 \in U_{\ddot{t}+1}$ and $m_2 \preceq k$. Then, $m_2 \in U'_{\ddot{t}+1}$. Further, either $[m_2 \prec k]$ or $[m_2 = k$ and $m_1 \prec j_1]$. Since $j \overset{R'}{\underset{\ddot{t}+1}{\not\longrightarrow}} m_1$, we have $m_1 \overset{R'}{\underset{\ddot{t}+1}{\not\longrightarrow}} m_2$. Let $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2$. Since

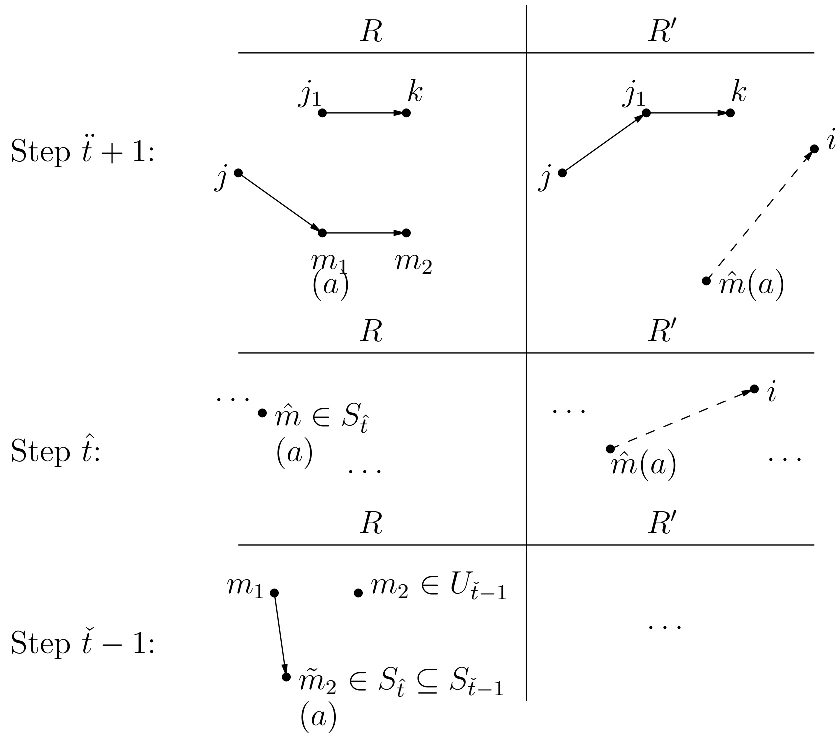
$m_2 \in U'_{\ddot{t}+1}$, we have $m'_2 \in U'_{\ddot{t}+1}$ and $m'_2 \prec m_2$. Then, $m'_2 \prec k$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.



$\boldsymbol{h'_{\ddot{t}+1}(m_1) \neq h_{\ddot{t}+1}(m_1)}$: Let $a \equiv h_{\ddot{t}+1}(m_1)$. By the *induction hypothesis*, since $h'_{\ddot{t}+1}(m_1) \neq a$, $m_1 \in CONN(i, R', \ddot{t})$. Thus, $m_1 \in CONN(i, R', \ddot{t}+1)$. Further, $m_1 \in S_{\ddot{t}+1}$. Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$. Suppose $m_1 \xrightarrow[\ddot{t}+1]{R} m_2$. Since $j \xrightarrow[\ddot{t}+1]{R} m_1$, we have $m_2 \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$ and $m_2 \prec k$.

Since $j \not\xrightarrow[\ddot{t}+1]{R'} \hat{m}$, $\hat{m} \in S'_{\ddot{t}+1}$. Since $h_{\ddot{t}+1}(\hat{m}) \neq a$, by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t}+1)$. So there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$. Then, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.

Now we consider the first $\check{t}$, which is between $\hat{t}$ and $\ddot{t}+1$, such that $h_{\check{t}}(m_1) = a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$ which contradicts $m_2 \in U_{\ddot{t}+1} \subseteq U_{\check{t}-1}$.



Stage 5) Next we consider the people who point at satisfied people whose pointees satisfied and have unsatisfied pointees, under R'. Particularly, consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ be such that $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} j_2 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}+1}$. Then, $j_1, j_2 \in S_{\ddot{t}+1}$.

By the preceding arguments, $j_1 \xrightarrow[\ddot{t}+1]{R} j_2 \xrightarrow[\ddot{t}+1]{R} k \in U_{\ddot{t}+1}$. Suppose $j \xrightarrow[\bar{t}]{R} m_1 \neq j_1$. Let $m_1 \xrightarrow[\ddot{t}+1]{R} m_2 \xrightarrow[\ddot{t}+1]{R} m_3$. We consider the following cases.

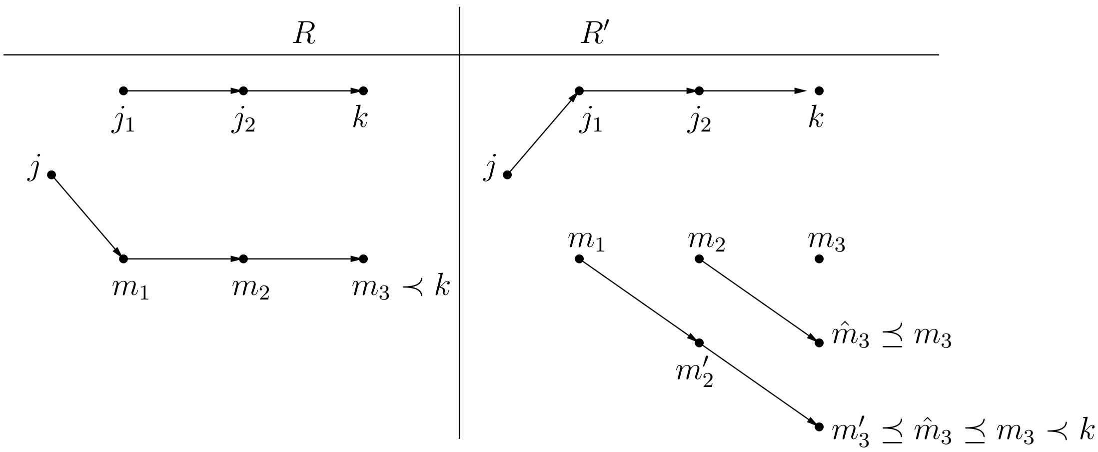
$\boldsymbol{h'_{\ddot{t}+1}(m_1) = h_{\ddot{t}+1}(m_1)}$: If $m_1 \in U_{\ddot{t}+1}$, then $m_1 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_1) = h'_{\ddot{t}+1}(m_1) = \omega(m_1)$. Then, $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_1 \in S_{\ddot{t}+1}$. Two sub-cases are as follows:

$\boldsymbol{h'_{\ddot{t}+1}(m_2) = h_{\ddot{t}+1}(m_2)}$: If $m_2 \in U_{\ddot{t}+1}$, then $m_2 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_2) = h'_{\ddot{t}+1}(m_2) = \omega(m_2)$. Then, $m_1 \xrightarrow[\ddot{t}+1]{R} U'_{\ddot{t}+1}$ and $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_2 \in S_{\ddot{t}+1}$.

Since $j \xrightarrow[\ddot{t}+1]{R} m_1 \neq j_1$, $m_3 \in U_{\ddot{t}+1}$. Further, $m_3 \in U'_{\ddot{t}+1}$ and either $[m_3 \prec k]$ or $[m_3 = k$ and $m_1 \prec j_1]$. Since, $j \not\xrightarrow[\ddot{t}+1]{R'} m_1$, then either,

(a) $m_1 \xrightarrow[\ddot{t}+1]{R'} m_2 \xrightarrow[\ddot{t}+1]{R'} m'_3 \neq m_3$: Since $m_3 \in U'_{\ddot{t}+1}$, $m'_3 \in U'_{\ddot{t}+1}$ and $m'_3 \prec m_3 \prec k$. This contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.

(b) $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2 \neq m_2$: Since $j \xrightarrow[\ddot{t}+1]{R'} j_1 \neq m_1$, we have $m'_2 \in S'_{\ddot{t}+1}$. Suppose $m_2 \xrightarrow[\ddot{t}+1]{R'} \hat{m}_3$ and $m'_2 \xrightarrow[\ddot{t}+1]{R'} m'_3$. Since $m_3 \in U'_{\ddot{t}+1}$, $\hat{m}_3 \in U'_{\ddot{t}+1}$ and $\hat{m}_3 \preceq m_3$. Since $m'_2 \in S'_{\ddot{t}+1}$, $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2$, and $\hat{m}_3 \in U'_{\ddot{t}+1}$, we have $m'_3 \in U'_{\ddot{t}+1}$ and $m'_3 \preceq \hat{m}_3$. Thus, $m'_3 \prec k$ which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.



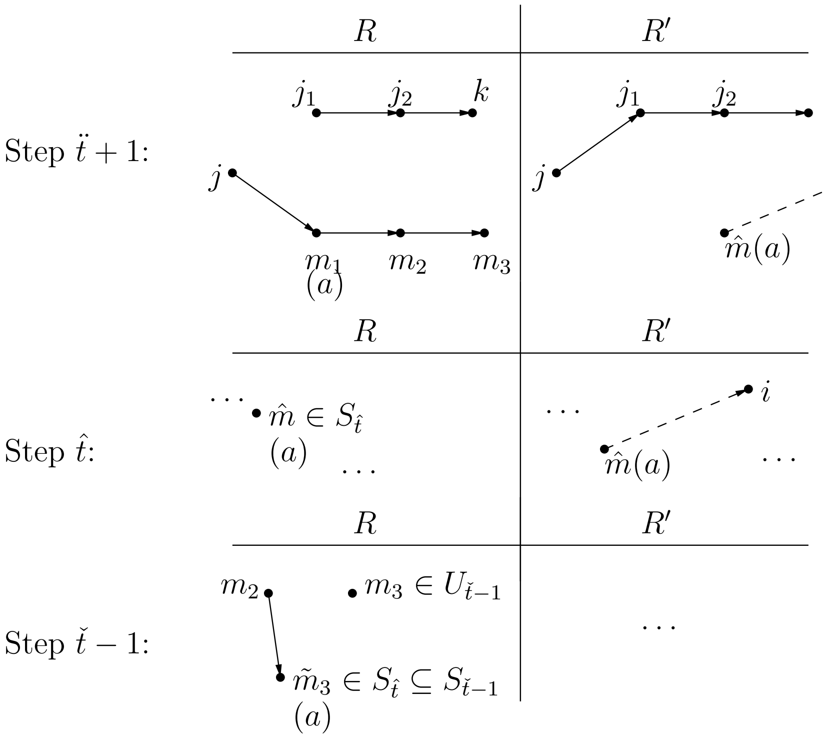
$\boldsymbol{h'_{\ddot{t}+1}(m_2) \neq h_{\ddot{t}+1}(m_2)}$: Let $a \equiv h_{\ddot{t}+1}(m_2)$. By the *induction hypothesis*, since $h'_{\ddot{t}+1}(m_2) \neq a$, we have $m_2 \in S_{\ddot{t}+1}$. Since $j \xrightarrow[\ddot{t}]{R} m_1$, $m_3 \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$.

Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$ and by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t} + 1)$.

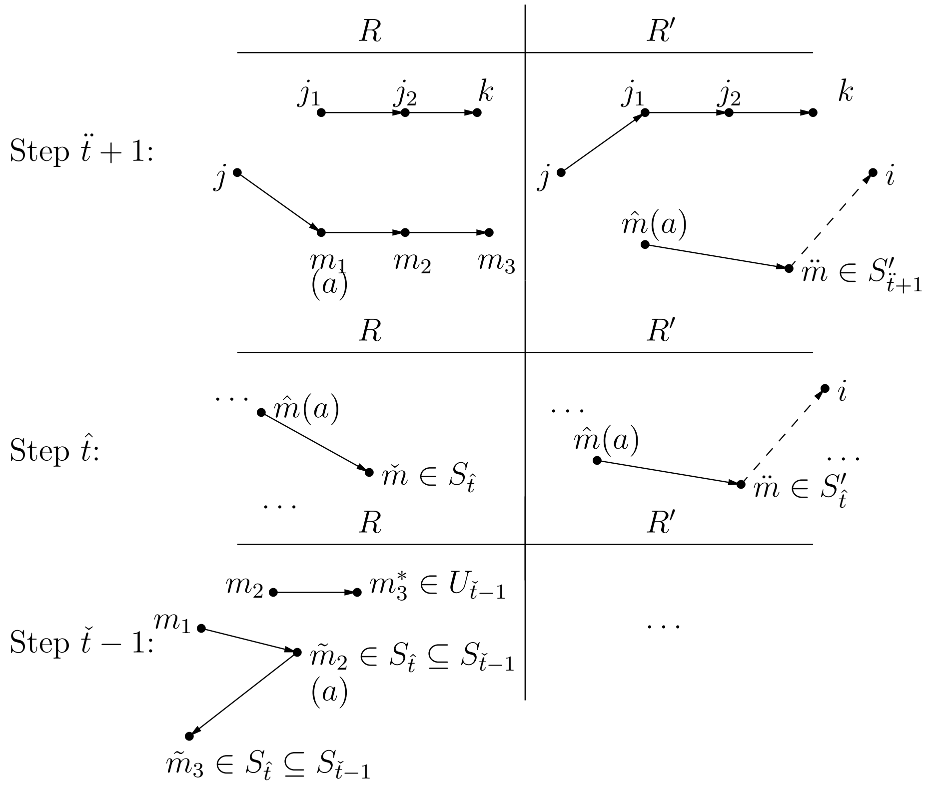
Since $a\ I_{m_1}\ h_{\ddot{t}+1}(m_1)$, and $j \not\xrightarrow[\ddot{t}+1]{R'} m_1$, we have that $m_1 \in S'_{\ddot{t}+1}$, $m_1 \xrightarrow[\ddot{t}+1]{R'} S'_{\ddot{t}+1}$, and $\hat{m} \in S'_{\ddot{t}+1}$.

Since $h_{\ddot{t}+1}(\hat{m}) \neq a$ and since there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.

Now consider the first $\check{t}$, which is between $\hat{t}$ and $\ddot{t}+1$, such that $h_{\check{t}}(m_2) = a$. Then, $m_2 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$ which contradicts $m_3 \in U_{\ddot{t}+1} \subseteq U_{\check{t}-1}$.



$h'_{\ddot{t}+1}(m_1) \not\sim h_{\ddot{t}+1}(m_1)$: Let $a \equiv h_{\ddot{t}+1}(m_1)$. Since $h'_{\ddot{t}+1}(m_1) \neq a$, $m_1 \in S_{\ddot{t}+1}$. Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$. Since $j \xrightarrow[\ddot{t}+1]{R'} j_1$, we have that $\hat{m} \in S'_{\ddot{t}+1}$ and $\hat{m} \xrightarrow[\ddot{t}+1]{R'} S'_{\ddot{t}+1}$. Since $h_{\ddot{t}+1}(\hat{m}) \neq a$, by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t}+1)$ and there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$. Since $\hat{m} \in S'_{\ddot{t}+1}$ and $p'_{\ddot{t}+1}(\hat{m}) = p'_{\hat{t}}(\hat{m})$, we have that $\hat{m} \in S'_{\hat{t}}$. This implies that $\hat{m} \in S_{\hat{t}}$ and $h_{\hat{t}}(\hat{m}) = a$. Since $\hat{m} \xrightarrow[\hat{t}]{R'} S'_{\hat{t}}$, then $\hat{m} \xrightarrow[\hat{t}]{R} S_{\hat{t}}$. And for each $\hat{\hat{t}} > \hat{t}$, we have $\hat{m} \xrightarrow[\hat{\hat{t}}]{R} S_{\hat{\hat{t}}}$. Now consider the first $\check{t}$, which is between $\hat{t}$ and $\ddot{t}+1$, such that $h_{\check{t}}(m_1) = a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1} \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$. However, if $m_2 \in U_{\ddot{t}+1} \subseteq U_{\check{t}}$, then $m_1 \xrightarrow[\ddot{t}+1]{R} U_{\ddot{t}+1} \subseteq U_{\check{t}}$ and if $m_2 \in S_{\ddot{t}+1}$, then $m_3 \in U_{\ddot{t}+1}$ and $m_1 \xrightarrow[\ddot{t}+1]{R} S_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R} U_{\ddot{t}+1}$. In either case, we have reached a contradiction.



Stage ...) Repeating this argument for the rest of the pointing phase we show (iii).

Now, we prove (v) for $\ddot{t}+1$. Suppose $j \in CONN(i,R,\ddot{t}+1) \setminus CONN(i,R',\ddot{t}+1)$. Then, there is $\{j_1, j_2, ..., j_r, i\} \subset N_{\ddot{t}+1} \subseteq N'_{\ddot{t}+1}$, such that $j \xrightarrow[\ddot{t}+1]{R} j_1 \xrightarrow[\ddot{t}+1]{R} j_2 \xrightarrow[\ddot{t}+1]{R} ... \xrightarrow[\ddot{t}+1]{R} j_r \xrightarrow[\ddot{t}+1]{R} i$. Since $j \notin CONN(i,R',\ddot{t}+1)$, by (iii), $j \xrightarrow[\ddot{t}+1]{R'} j_1$. Then, $j_1 \notin CONN(i,R',\ddot{t}+1)$. Again, by (iii), $j_1 \xrightarrow[\ddot{t}+1]{R'} j_2$ and $j_2 \notin CONN(i,R',\ddot{t}+1)$. Repeating the argument r times, $j_r \notin CONN(i,R',\ddot{t}+1)$. By (iii), $j_r \xrightarrow[\ddot{t}+1]{R'} i$, and this contradicts $j \notin CONN(i,R',\ddot{t}+1)$.

Finally, we prove (iv) for Step $\ddot{t}+1$. We show that for each $j \in N'_{\ddot{t}+1} \setminus CONN(i,R',\ddot{t}+1)$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$. By (iii) each *trading cycle* that does not involve people connected to i under R' is also a *trading cycle* under R. Therefore, for each $j \in N'_{\ddot{t}+1} \setminus CONN(i,R',\ddot{t}+1)$, $h'_{\ddot{t}+2}(j) = h_{\ddot{t}+2}(j)$. Moreover, for each $j \in CONN(i,R',\ddot{t}+1)$, $h'_{\ddot{t}+2}(j) = h'_{\ddot{t}+1}(j)$. $\diamond$

If $t' \leq t$, by *pre-trade inclusion*, and the $\underline{t}$ *equality* lemma, $O_t \subseteq O'_t$ and $O'_t \subseteq O'_{t'}$. Thus, $\alpha(i) \in O'_{t'}$. Since i is part of a *trading cycle* at Step t', he is assigned one of his most preferred objects in $O'_{t'}$ which is uniquely $\alpha(i)$. Thus, $\alpha'(i) = \alpha(i)$.

Suppose not, then $t' > t$. To show that $\alpha(i) = \alpha'(i)$ we first prove the following claim.

Claim 2. ***(Post-trade inclusion)*** *For each* $\ddot{t} \in \{t.., t'\}$,

(i) $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$, $N_{\ddot{t}} \subseteq N'_{\ddot{t}}$, $O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i, R', \ddot{t}-1))$, *and* $N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1)$,

(ii) $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ *and* $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1)$,

(iii) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$, *and*

(iv) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$.

The proof of this claim is similar to that of *pre-trade inclusion*. We have provided it in the appendix.

Suppose $\alpha'(i) \neq \alpha(i)$. Since i is assigned $\alpha(i)$ under R, there is $\tilde{t}$ such that $h_{\tilde{t}+1}(i) = \alpha(i)$. By *post-trade inclusion*, $\tilde{t} < t'$. Since $\tau(R'_i, O_{\tilde{t}}) = \{\alpha(i)\}$, we have $i \xrightarrow[\tilde{t}]{R'} j \in N'_{\tilde{t}}$ such that $h'_{\tilde{t}}(j) = \alpha(i)$. Since $\alpha'(i) \neq \alpha(i)$, $h'_{\tilde{t}+1}(i) \neq \alpha(i)$. Thus $j \notin CONN(i, R', \tilde{t}+1)$. Thus by *post-trade inclusion*, $h_{\tilde{t}}(j) = h'_{\tilde{t}}(j) = \alpha(i)$. Since $j \notin CONN(i, R', \tilde{t})$, we have $j \xrightarrow[\tilde{t}]{R'} j_1(\neq i) \xrightarrow[\tilde{t}]{R'} j_2(\neq i) \dots \xrightarrow[\tilde{t}]{R'} j_r(\neq i)$. Again, by *post-trade inclusion*, $j \xrightarrow[\tilde{t}]{R} j_1(\neq i) \xrightarrow[\tilde{t}]{R} j_2(\neq i) \dots \xrightarrow[\tilde{t}]{R} j_r(\neq i)$. This contradicts $h_{\tilde{t}}(i) = \alpha(i)$. ♣

We are now ready to show that $TC^{\prec}$ is *strategy-proof.*

Proposition 5. *For priority order* $\prec$, $TC^{\prec}(R, \omega)$ *is strategy-proof.*

Proof: Suppose that $TC^{\prec}$ is not *strategy-proof.* Then, there is $(R, \omega) \in \mathcal{R}^N \times A$, $i \in N$ and $R'_i \in \mathcal{R}$ such that $TC^{\prec}(R'_i, R_{-i}, \omega)(i)\ P_i\ TC^{\prec}(R, \omega)(i)$. Let $\alpha \equiv TC^{\prec}(R, \omega)$ and $\alpha' \equiv TC^{\prec}(R'_i, R_{-i}, \omega)$. By the *invariance* lemma, we only need to consider R'_i such that $I(\alpha'(i), R_i) = \{\alpha'(i)\}$. Otherwise, there is $R_i^{\alpha'(i)^{\uparrow}} \in R$ such that $TC^{\prec}(R_i^{\alpha(i)^{\uparrow}}, R_{-i}, \omega)(i) = \alpha'(i)$ and thus, $TC^{\prec}(R_i^{\alpha'(i)^{\uparrow}}, R_{-i}, \omega)(i)\ P_i\ \alpha(i)$.

Define $t, t', \bar{t}$, and $\underline{t}$ as in the proof of the *invariance* lemma. Since $\alpha'(i) \neq \omega(i), \alpha'(i)\ P'_i\ \omega(i)$ and for each $\ddot{t} \leq t'$, $i \in U'_{\ddot{t}}$. We consider the following cases.

Case 1: $\underline{t} = \bar{t} \leq t'$. In this case, $i \in S_{\bar{t}}$. That is, $\omega(i) \in \tau(R_i, O_{\bar{t}})$. By the $\underline{t}$ *equality* lemma, $O_{\bar{t}} = O'_{\bar{t}}$. Since $\alpha'(i) \in O'_{\bar{t}}$, $\alpha'(i) \in O_{\bar{t}}$. Thus, $\omega(i)\ R_i\ \alpha'(i)$ and by *individual rationality*, $\alpha(i)\ R_i\ \alpha'(i)$.

Case 2: $\underline{t} = t' < t$. By the $\underline{t}$ *equality* lemma, $O_{t'} = O'_{t'}$, $N_{t'} = N'_{t'}$, and for each $j \in N'_{t'} \setminus \{i\}$, $p_{t'}(j) = p'_{t'}(j)$ and $h_{t'}(j) = h'_{t'}(j)$. Since i trades under R', $\{h'_{t'+1}(i)\} = \{\alpha'(i)\} = \tau(R'_i, O'_{t'})$. Then, i leaves with $\alpha'(i)$. Therefore, there is $\{j_1, j_2, \ldots, j_3\} \subseteq N'_{\underline{t}}$ such that $j_1 \xrightarrow[\underline{t}]{R'} j_2 \xrightarrow[\underline{t}]{R'} j_3 \ldots \xrightarrow[\underline{t}]{R'} j_r \xrightarrow[\underline{t}]{R'} i$ and $h'_{\underline{t}}(j_1) = \alpha'(i)$. Then, by the $\underline{t}$ *equality* lemma, $j_1 \xrightarrow[\underline{t}]{R} j_2 \xrightarrow[\underline{t}]{R} j_3 \ldots \xrightarrow[\underline{t}]{R} j_r \xrightarrow[\underline{t}]{R} i$ and $h_{\underline{t}}(j_1) = \alpha'(i)$. By *persistence*, $h_{t+1}(i)\ R_i\ \alpha'(i)$.

Case 3: $\underline{t} = t \le t'$. Since $h_{t+1}(i) \in \tau(R_i, O_t)$ and $h_{t+1}(i)\ I_i\ \alpha(i)$, $\alpha(i) \in \tau(R_i, O_t)$. Since $\alpha'(i) \in O'_t$ and by by the $\underline{t}$ *equality* lemma $O'_t = O_t$ we have $\alpha'(i) \in O_t$. Thus, $\alpha(i)\ R_i\ \alpha'(i)$. ■

6 Generality of our model

In this section, we show that the model that we have studied is general enough to include the problems where there may or may not be a private endowment in addition to a social endowment (Hylland and Zeckhauser 1979, Abdulkadiroğlu and Sönmez 1999).

Let $\tilde{O}$ be a set of objects and $\tilde{N}$ be a set of people. Let $\varnothing \notin \tilde{O}$ be the **null object**. The **private endowment, $\tilde{\omega} : \tilde{N} \to \tilde{O} \cup \{\varnothing\}$**, is such that for each $i, j \in \tilde{N}$, $\tilde{\omega}(i) \neq \tilde{\omega}(j)$ unless $\tilde{\omega}(i) = \varnothing$. Let $\tilde{\mathcal{R}}$ be the set of preference relations over $\tilde{O}$. Let $\tilde{R} \in \tilde{\mathcal{R}}^{\tilde{N}}$. The tuple $(\tilde{O}, \tilde{N}, \tilde{\omega}, \tilde{R})$ defines a problem. We show how this problem can be encoded as a problem in our original model without *social endowments*.

Define (O, N, ω, R) as follows. For each $a \in \tilde{O} \setminus \tilde{\omega}(\tilde{N})$, we introduce i_a, a "dummy person" with degenerate preferences, $R_{i_a} = \overline{I_0}$. For each $i \in \tilde{N}$ such that $\tilde{\omega}(i) = \varnothing$, we introduce d_i, a "dummy object" which every person considers to be worse than any object in $\tilde{O}$. For each person in $\tilde{N}$, his preferences over $\tilde{O}$ are kept the same. That is,

$$\begin{aligned} O &\equiv \tilde{O} \cup \{d_i : \text{for each } i \in \tilde{N} \text{ such that } \tilde{\omega}(i) = \varnothing\}, \\ N &\equiv \tilde{N} \cup \{i_a : \text{for each } a \in \tilde{O} \setminus \tilde{\omega}(\tilde{N})\}, \\ \text{For each } i \in N, \omega(i) &\equiv \begin{cases} \tilde{\omega}(i) & \text{if } i \in \tilde{N} \text{ and } \tilde{\omega}(i) \neq \varnothing \\ d_i & \text{if } i \in \tilde{N} \text{ and } \tilde{\omega}(i) = \varnothing \\ a & \text{if } i = i_a, \text{ and} \end{cases} \end{aligned}$$

$R \in \mathcal{R}^N$ is such that for each $i \in \tilde{N}$, $R_i|_{\tilde{O}} = \tilde{R}_i|_{\tilde{O}}$, and for each $d_j \in O \setminus \tilde{O}$ and each $a \in \tilde{O}, a\ P_i\ d_j$.

We also point out that the *top cycles rules* described in this paper can be generalized to problems where each object is associated with an "inheritance hierarchy" and each person may be endowed with any number of objects (Pápai 2000).

Remark 4. (Two-sided matching) The "two-sided matching" model is closely related to the one we study here. Dropping the assumption that people are never indifferent between alternatives has a striking effect on this model as well. The "deferred acceptance" algorithm holds a place in the literature on this model similar to the one held by the *top trading cycles* algorithm for our model. The key axioms for two-sided matching are "stability" and *efficiency*, both of which are satisfied by the *deferred acceptance* algorithm. Like the *top trading cycles* algorithm, the *deferred acceptance* algorithm can be adapted to preserve both these axioms in the presence of indifference (Erdil and Ergin 2006). However, this adaptation does not preserve "one-sided" *strategy-proofness*, which means that an application of this algorithm to our model does not satisfy *strategy-proofness*. ∘

7 Conclusion

Through Proposition 5, we show that *strategy-proofness, Pareto-efficiency,* and *individual rationality* are compatible. But this leaves open the question of what other rules satisfy these properties. We know that any such rule selects from the *core* when it is non-empty (Ma 1994), but it is not yet clear what it can select when the *core* is empty. It is easy to show that the three axioms are independent. While there are rules other than *top cycles* rules that satisfy them, the only such rules that we are aware of are minimal variation of *top cycles* rules. For instance, the tie-breaking scheme can be changed when two satisfied candidate pointees point at the same unsatisfied person. It is still unclear if more substantial departures are possible.

Appendices

A Proof of the "post-trade inclusion" claim

Claim 2: (Post-trade inclusion)
For each $\ddot{t} \in \{t.., t'\}$,

(i) $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$, $N_{\ddot{t}} \subseteq N'_{\ddot{t}}$, $O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i, R', \ddot{t}-1))$, *and* $N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1)$,

(ii) $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ *and* $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1)$,

(iii) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$, *and*

(iv) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$.

Proof: Let $\ddot{t} = t+1$. First, we prove statements (i) and (ii) for $t+1$. At t, i is a member of a *trading cycle* under R, but not under R'. By *pre-trade inclusion*, each trading cycle that does not involve people connected to i under R' is also a trading cycle under R. In addition, for each $j \in N'_t \setminus CONN(i, R', t)$, $h_{t+1}(j) = h'_{t+1}(j)$. Thus, if $C \in N_t$ is a trading cycle under R but not under R', then $C \subset CONN(i, R', t)$. Therefore, at Step $t+1$, $O_{t+1} \subset O'_{t+1}$, $O'_{t+1} \setminus O_{t+1} \subseteq h_{t+1}(CONN(i, R', t))$, $N_{t+1} \subset N'_{t+1}$, and $N'_{t+1} \setminus N_{t+1} \subseteq CONN(i, R', t)$. Further, $S'_{t+1} \subset S_{t+1}$ and $S_{t+1} \setminus S'_{t+1} \subseteq CONN(i, R', t)$.

We now prove (iii) for $t+1$, by following the progression of the pointing phase.

Stage 1) We first consider people whos pointee in Step t remains in N'_{t+1} and holds the same object. In particular, we consider $j \in N'_{t+1} \setminus CONN(i, R', t+1)$ such that $j \xrightarrow[t]{R'} k \in N'_{t+1}$ and $h'_{t+1}(k) = h'_t(k)$. Then, $j \xrightarrow[t+1]{R'} k$. By *pre-trade inclusion*, $j \xrightarrow[t]{R} k$ and $h_{t+1}(k) = h_t(k) = h'_t(k)$. Thus, by (ii), $j \xrightarrow[t+1]{R} k$.

Stage 2) Now we consider people who have a unique most preferred object. For each $j \in N'_{t+1} \setminus CONN(i, R', t+1)$, if $\tau(R_j, O'_{t+1}) = \{a\}$, then by *pre-trade inclusion*, $h^{-1}_{t+1}(a) = h'^{-1}_{t+1}(a) \notin CONN(i, R', t+1)$. Thus, $a \in O_{t+1}$ and so $p_{t+1}(j) = p'_{t+1}(j)$.

Stage 3) Next, we consider the people with unsatisfied pointees under R'. In particular, we consider $j \in N'_{t+1} \setminus CONN(i, R', t+1)$ such that $j \xrightarrow[t+1]{R'} k \in U'_{t+1}$. Since $j \notin CONN(i, R', t+1)$, $k \notin CONN(i, R', t+1)$. Since $k \in U'_{t+1}$ and $S_{t+1} \setminus S'_{t+1} \subset CONN(i, R', t+1)$, $k \in U_{t+1}$. Further, $h_{t+1}(k) = h'_{t+1}(k) = \omega(k)$. Suppose $j \xrightarrow[t+1]{R} m \neq k$. Then, $m \in U_{t+1} \subset U'_{t+1}$ and so $h_{t+1}(m) = h'_{t+1}(m) = \omega(m)$ and $m \prec k$. This contradicts $j \xrightarrow[t+1]{R'} k$.

Stage 4) We now consider the people who point at satisfied people with unsatisfied pointees, under R'. In particular, we consider $j \in N'_{t+1} \setminus CONN(i, R', t+1)$ such that $j \xrightarrow[t+1]{R'} j_1 \in S'_{t+1} \xrightarrow[t+1]{R'} k \in U'_{t+1}$. Then, by (ii), $j_1 \in S_{t+1}$.

By the preceding arguments, $j_1 \xrightarrow[t+1]{R} k$ and $k \in U_{t+1}$. Suppose $j \xrightarrow[t+1]{R} m_1 \neq j_1$. We consider the following cases.

$\boldsymbol{h'_{t+1}(m_1) = h_{t+1}(m_1)}$: If $m_1 \in U_{t+1}$, then $m_1 \in U'_{t+1}$ and $h_{t+1}(m_1) = h'_{t+1}(m_1) = \omega(m_1)$. Then, $j \xrightarrow[t+1]{R'} m_1$, which contradicts $j \xrightarrow[t+1]{R'} j_1 \in S'_{t+1}$. Thus, $m_1 \in S_{t+1}$.

Suppose $m_1 \xrightarrow[t+1]{R} m_2$. Since $j \xrightarrow[t+1]{R} m_1$ and $k \in U_{t+1}$ and $m_2 \preceq k$. Then, $m_2 \in U'_{t+1}$. Further, either $[m_2 \prec k]$ or $[m_2 = k$ and $m_1 \prec j_1]$. Since $j \not\xrightarrow[t+1]{R'} m_1$, we have $m_1 \not\xrightarrow[t+1]{R'} m_2$. Let $m_1 \xrightarrow[t+1]{R'} m'_2$. Since $m_2 \in U'_{t+1}$, we have $m'_2 \notin S'_{t+1}$ and $m'_2 \prec m_2$. Then, $m'_2 \prec k$, which contradicts $j \xrightarrow[t+1]{R'} j_1$.

$\boldsymbol{h'_{t+1}(m_1) \neq h_{t+1}(m_1)}$: Let $a \equiv h_{t+1}(m_1)$. By *pre-trade inclusion*, since $h'_{t+1}(m_1) \neq a$, $m_1 \in CONN(i, R', t)$. Thus, $m_1 \in CONN(i, R', t+1)$. Further, $m_1 \in S_{t+1}$. Since $O_{t+1} \subseteq O'_{t+1}$, there is $\hat{m} \in N'_{t+1}$ such that $h'_{t+1}(\hat{m}) = a$. Suppose $m_1 \xrightarrow[t+1]{R} m_2$. Since $j \xrightarrow[t+1]{R} m_1$, we have $m_2 \in U_{t+1} \subseteq U'_{t+1}$ and $m_2 \prec k$.

Since $j \not\xrightarrow[t+1]{R'} \hat{m}$, $\hat{m} \in S'_{t+1}$.

Since $h_{t+1}(\hat{m}) \neq a$, by *pre-trade inclusion*, $\hat{m} \in CONN(i, R', t+1)$. So there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.

Now we consider the first $\check{t}$, which is between $\hat{t}$ and $t+1$, such that $h_{\check{t}}(m_1) = a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$, which contradicts $m_2 \in U_{t+1} \subseteq U_{\check{t}-1}$.

Stage 5) Now we consider the people who point at satisfied people whose pointees are satisfied people with unsatisfied pointees under R'. Particularly, we consider $j \in N'_{t+1} \setminus CONN(i, R', t+1)$ such that $j \xrightarrow[t+1]{R'} j_1 \in S'_{t+1} \xrightarrow[t+1]{R'} j_2 \in S'_{t+1} \xrightarrow[t+1]{R'} k \in U'_{t+1}$. Then, by (ii), $j_1, j_2 \in S_{t+1}$.

By the preceding arguments, $j_1 \xrightarrow[t+1]{R} j_2 \xrightarrow[t+1]{R} k \in U_{t+1}$. Suppose $j \xrightarrow[\bar{t}]{R} m_1 \neq j_1$. Let $m_1 \xrightarrow[t+1]{R} m_2 \xrightarrow[t+1]{R} m_3$. We consider the following cases.

$\boldsymbol{h'_{t+1}(m_1) = h_{t+1}(m_1)}$: If $m_1 \in U_{t+1}$, then $m_1 \in U'_{t+1}$ and $h_{t+1}(m_1) = h'_{t+1}(m_1) = \omega(m_1)$. Then, $j \xrightarrow[t+1]{R'} m_1$, which contradicts $j \xrightarrow[t+1]{R'} j_1 \in S'_{t+1}$. Thus, $m_1 \in S_{t+1}$. Two sub-cases are as follows:

$\boldsymbol{h'_{t+1}(m_2) = h_{t+1}(m_2)}$: If $m_2 \in U_{t+1}$, then $m_2 \in U'_{t+1}$ and $h_{t+1}(m_2) = h'_{t+1}(m_2) = \omega(m_2)$. Then, $m_a \xrightarrow[t+1]{R'} U'_{t+1}$ and $j \xrightarrow[t+1]{R'} m_1$, which contradicts $j \xrightarrow[t+1]{R'} j_1 \in S'_{t+1}$. Thus, $m_2 \in S_{t+1}$.

Since $j \xrightarrow[t+1]{R} m_1 \neq j_1$, $m_3 \in U_{t+1}$. Further, $m_3 \in U'_{t+1}$ and either $[m_3 \prec k]$ or $[m_3 = k$ and $m_1 \prec j_1]$. Since, $j \not\xrightarrow[t+1]{R'} m_1$, then either,

(a) $m_1 \xrightarrow[t+1]{R'} m_2 \xrightarrow[t+1]{R'} m'_3 \neq m_3$: Since $m_3 \in U'_{t+1}$, $m'_3 \in U'_{t+1}$ and $m'_3 \prec m_3 \prec k$. This contradicts $j \xrightarrow[t+1]{R'} j_1$.

(b) $m_1 \xrightarrow[t+1]{R'} m'_2 \neq m_2$: Since $j \xrightarrow[t+1]{R'} j_1 \neq m_1$, we have $m'_2 \in S'_{t+1}$. Suppose $m_2 \xrightarrow[t+1]{R'} \hat{m}_3$ and $m'_2 \xrightarrow[t+1]{R'} m'_3$. Since $m_3 \in U'_{t+1}$, $\hat{m}_3 \in U'_{t+1}$ and $\hat{m}_3 \preceq m_3$. Since $m'_2 \in S'_{t+1}$, $m_1 \xrightarrow[t+1]{R'} m'_2$, and $\hat{m}_3 \in U'_{t+1}$, we have $m'_3 \in U'_{t+1}$ and $m'_3 \prec \hat{m}_3$. Thus, $m'_3 \prec k$ which contradicts $j \xrightarrow[t+1]{R'} j_1$.

$\boldsymbol{h'_{t+1}(m_2) \neq h_{t+1}(m_2)}$: Let $a \equiv h_{t+1}(m_2)$. By *pre-trade inclusion*, since $h'_{t+1}(m_2) \neq a$, we have $m_2 \in S_{t+1}$. Since $j \xrightarrow[\ddot{t}]{R} m_1$, $m_3 \in U_{t+1} \subseteq U'_{t+1}$.
Since $O_{t+1} \subseteq O'_{t+1}$, there is $\hat{m} \in N'_{t+1}$ such that $h'_{t+1}(\hat{m}) = a$ and by *pre-trade inclusion*, $\hat{m} \in CONN(i, R', t+1)$.
Since $a\ I_{m_1}\ h_{t+1}(m_1)$, and $j \not\xrightarrow[t+1]{R'} m_1$, we have that $m_1 \in S'_{t+1}$, $m_1 \xrightarrow[t+1]{R'} S'_{t+1}$, and $\hat{m} \in S'_{t+1}$.
Since $h_{t+1}(\hat{m}) \neq a$ and since there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.
Now consider the first $\check{t}$, which is between $\hat{t}$ and $t+1$, such that $h_{\check{t}}(m_2) = a$. Then, $m_2 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$ which contradicts $m_3 \in U_{\ddot{t}+1} \subseteq U_{\check{t}-1}$.

$\boldsymbol{h'_{t+1}(m_1) \not\Rightarrow h_{t+1}(m_1)}$: Let $a \equiv h_{t+1}(m_1)$. Subce $h'_{t+1}(m_1) \neq a$, $m_1 \in S_{t+1}$. Since $O_{t+1} \subseteq O'_{t+1}$, there is $\hat{m} \in N'_{t+1}$ such that $h'_{t+1}(\hat{m}) = a$. Since $j \xrightarrow[t+1]{R'} j_1$, we have that $\hat{m} \in S'_{t+1}$ and $\hat{m} \xrightarrow[t+1]{R'} S'_{t+1}$. Since $h_{t+1}(\hat{m}) \neq a$, by *pre-trade inclusion*, $\hat{m} \in CONN(i, R', t+1)$ and there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$. Since $\hat{m} \in S'_{t+1}$ and $p'_{\hat{t}}(\hat{m}) = p'_{t+1}(\hat{m})$, we have that $\hat{m} \in S'_{\hat{t}}$. This implies that $\hat{m} \in S_{\hat{t}}$ and $h_{\hat{t}}(\hat{m}) = a$. Since $\hat{m} \xrightarrow[\hat{t}]{R'} S'_{\hat{t}}$, then $\hat{m} \xrightarrow[\hat{t}]{R} S_{\hat{t}}$. And for each $\hat{\hat{t}} > \hat{t}$, we have $\hat{m} \xrightarrow[\hat{\hat{t}}]{R} S_{\hat{\hat{t}}}$. Now consider the first $\check{t}$, which is between $\hat{t}$ and $t+1$, such that $h_{\check{t}}(m_1) = a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1} \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$. However, if $m_2 \in U_{t+1} \subseteq U_{\check{t}}$, then $m_1 \xrightarrow[t+1]{R} U_{t+1} \subseteq U_{\check{t}}$ and if $m_2 \in S_{t+1}$, then $m_3 \in U_{t+1}$ and $m_1 \xrightarrow[t+1]{R} S_{t+1} \xrightarrow[t+1]{R} U_{t+1}$. In either case, we have reached a contradiction.

Stage ...) Repeating this argument for the rest of the pointing phase we show (iii).

Now, we prove (iv) for $t+1$. That is, we show that for each $j \in N'_{t+1} \setminus CONN(i, R', t+1)$, $h_{t+1}(j) = h'_{t+1}(j)$. Note that since at $t+1 < t'$, i is not part of any *trading cycle* under R'. Thus is, no *trading cycle* under R' involves people connected to i under R'. That is for each trading cycle $C' \subset N'_{t+1}$, $CONN(i, R', t+1) \cap C' = \varnothing$. By (iii), each *trading cycle* that does not involve people connected to i under R' is also a *trading cycle* under R. Therefore, for each $j \in N'_{t+1} \setminus CONN(i, R', t+1)$, $h'_{t+2}(j) = h_{t+2}(j)$. Moreover, for each $j \in CONN(i, R', t+1)$, $h'_{t+2}(j) = h'_{t+1}(j)$.

As an **induction hypothesis**, suppose that for some $\ddot{t} \in \{t, ..., t'-1\}$,

(i) $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$, $O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i, R', \ddot{t}-1))$, and $N_{\ddot{t}} \subseteq N'_{\ddot{t}}$, $N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1)$,

(ii) $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ and $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1)$,

(iii) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$, and

(iv) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $h_{\ddot{t}}(j) = h'_{\ddot{t}}(j)$, and

Now we prove that these statements are true of $\ddot{t}+1$. To prove (i) and (ii) for $\ddot{t}+1$ note that by (iv) and (v) of the *induction hypothesis*, if $C \in N_{\ddot{t}}$ is a *trading cycle* under R and is not a *trading cycle* under R', then $C \subseteq CONN(i, R', \ddot{t})$. Thus, at Step $\ddot{t}+1$, we have statements (i) and (ii).

We now prove (iii), for $\ddot{t}+1$, by following the progression of the pointing phase just as in the case of $t+1$.

Stage 1) At the beginning of the pointing phase we consider people who were pointing at someone who remains in $N'_{\ddot{t}+1}$ and holds the same object. In particular, we consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}]{R'} k \in N'_{\ddot{t}}$ and $h'_{\ddot{t}+1}(k) = h'_{\ddot{t}}(k)$. Then, $j \xrightarrow[\ddot{t}+1]{R'} k$. By the *induction hypothesis*, $j \xrightarrow[\ddot{t}]{R} k$ and $h_{\ddot{t}+1}(k) = h_{\ddot{t}}(k) = h'_{\ddot{t}}(k)$. Thus $j \xrightarrow[\ddot{t}+1]{R} k$.

Stage 2) Now we consider people who have a unique most preferred object. For each $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$, if $\tau(R_j, O'_{\ddot{t}+1}) = \{a\}$, then by the *induction hypothesis*, $h^{-1}_{\ddot{t}+1}(a) = h'^{-1}_{\ddot{t}+1}(a) \notin CONN(i, R', \ddot{t}+1)$. Thus, $a \in O_{\ddot{t}+1}$ and so $p_{\ddot{t}+1}(j) = p'_{\ddot{t}+1}(j)$.

Stage 3) Next, we consider the people with unsatisfied pointees under R'. In particular, $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}}$. Since $j \notin CONN(i, R', \ddot{t}+1)$, $k \notin CONN(i, R', \ddot{t}+1)$. Since $k \in U'_{\ddot{t}+1}$ and $S_{\ddot{t}+1} \setminus S'_{\ddot{t}+1} \subseteq CONN(i, R', \ddot{t}+1)$, $k \in U_{\ddot{t}+1}$. Further, $h_{\ddot{t}+1}(k) = h'_{\ddot{t}+1}(k) = \omega(k)$. Suppose $j \xrightarrow[\ddot{t}+1]{R} m \neq k$. Then, $m \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$ and so $h_{\ddot{t}+1}(m) = h'_{\ddot{t}+1}(m) = \omega(m)$ and $m \prec k$. This contradicts $j \xrightarrow[\ddot{t}+1]{R'} k$.

Stage 4) We now consider the people who point at satisfied people with unsatisfied pointees, under R'. In particular, we consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}+1}$. Then, by (ii), $j_1 \in S_{\ddot{t}+1}$.

By the preceding arguments, $j_1 \xrightarrow[\ddot{t}+1]{R} k$ and $k \in U_{\ddot{t}+1}$. Suppose $j \xrightarrow[\ddot{t}+1]{R} m_1 \neq j_1$. We consider the following two cases.

$\boldsymbol{h'_{\ddot{t}+1}(m_1) = h_{\ddot{t}+1}(m_1)}$: If $m_1 \in U_{\ddot{t}+1}$, then $m_1 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_1) = h'_{\ddot{t}+1}(m_1) = \omega(m_1)$.

Then, $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_1 \in S_{\ddot{t}+1}$. Suppose $m_1 \xrightarrow[\ddot{t}+1]{R} m_2$. Since $j \xrightarrow[\ddot{t}+1]{R} m_1$ and $k \in U_{\ddot{t}+1}$, then $m_2 \in U_{\ddot{t}+1}$ and $m_2 \preceq k$. Then, $m_2 \in U'_{\ddot{t}+1}$. Further, either $[m_2 \prec k]$ or $[m_2 = k$ and $m_1 \prec j_1]$. Since $j \not\xrightarrow[\ddot{t}+1]{R'} m_1$, we have $m_1 \not\xrightarrow[\ddot{t}+1]{R'} m_2$. Let $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2$. Since $m_2 \in U'_{\ddot{t}+1}$, we have $m'_2 \in U'_{\ddot{t}+1}$ and $m'_2 \prec m_2$. Then, $m'_2 \prec k$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.

$\boldsymbol{h'_{\ddot{t}+1}(m_1) \neq h_{\ddot{t}+1}(m_1)}$: Let $a \equiv h_{\ddot{t}+1}(m_1)$. By the *induction hypothesis*, since $h'_{\ddot{t}+1}(m_1) \neq a$,

$m_1 \in CONN(i, R', \ddot{t})$. Thus, $m_1 \in CONN(i, R', \ddot{t}+1)$. Further, $m_1 \in S_{\ddot{t}+1}$. Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$.

Suppose $m_1 \xrightarrow[\ddot{t}+1]{R} m_2$. Since $j \xrightarrow[\ddot{t}+1]{R} m_1$, we have $m_2 \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$ and $m_2 \prec k$.

Since $j \not\xrightarrow[\ddot{t}+1]{R'} \hat{m}$, $\hat{m} \in S'_{\ddot{t}+1}$.

Since $h_{\ddot{t}+1}(\hat{m}) \neq a$, by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t}+1)$. So there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$. Then, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.

Now we consider the first $\check{t}$, which is between $\hat{t}$ and $\ddot{t}+1$, such that $h_{\check{t}}(m_1) = a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$ which contradicts $m_2 \in U_{\ddot{t}+1} \subseteq U_{\check{t}-1}$.

Stage 5) Next we consider the people who point at satisfied people whose pointees satisfied and have unsatisfied pointees, under R'. Particularly, consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ be such that $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} j_2 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}+1}$. Then, $j_1, j_2 \in S_{\ddot{t}+1}$.

By the preceding arguments, $j_1 \xrightarrow[\ddot{t}+1]{R} j_2 \xrightarrow[\ddot{t}+1]{R} k \in U_{\ddot{t}+1}$. Suppose $j \xrightarrow[\ddot{t}]{R} m_1 \neq j_1$. Let $m_1 \xrightarrow[\ddot{t}+1]{R} m_2 \xrightarrow[\ddot{t}+1]{R} m_3$. We consider the following cases.

$\boldsymbol{h'_{\ddot{t}+1}(m_1) = h_{\ddot{t}+1}(m_1)}$: If $m_1 \in U_{\ddot{t}+1}$, then $m_1 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_1) = h'_{\ddot{t}+1}(m_1) = \omega(m_1)$.

Then, $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_1 \in S_{\ddot{t}+1}$.

Two sub-cases are as follows:

$\boldsymbol{h'_{\ddot{t}+1}(m_2) = h_{\ddot{t}+1}(m_2)}$: If $m_2 \in U_{\ddot{t}+1}$, then $m_2 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_2) = h'_{\ddot{t}+1}(m_2) = \omega(m_2)$. Then, $m_1 \xrightarrow[\ddot{t}+1]{R} U'_{\ddot{t}+1}$ and $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_2 \in S_{\ddot{t}+1}$.

Since $j \xrightarrow[\ddot{t}+1]{R} m_1 \neq j_1$, $m_3 \in U_{\ddot{t}+1}$. Further, $m_3 \in U'_{\ddot{t}+1}$ and either $[m_3 \prec k]$ or $[m_3 = k$ and $m_1 \prec j_1]$. Since, $j \not\xrightarrow[\ddot{t}+1]{R'} m_1$, then either,

(a) $m_1 \xrightarrow[\ddot{t}+1]{R'} m_2 \xrightarrow[\ddot{t}+1]{R'} m'_3 \neq m_3$: Since $m_3 \in U'_{\ddot{t}+1}$, $m'_3 \in U'_{\ddot{t}+1}$ and $m'_3 \prec m_3 \prec k$. This contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.

(b) $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2 \neq m_2$: Since $j \xrightarrow[\ddot{t}+1]{R'} j_1 \neq m_1$, we have $m'_2 \in S'_{\ddot{t}+1}$.

Suppose $m_2 \xrightarrow[\ddot{t}+1]{R'} \hat{m}_3$ and $m'_2 \xrightarrow[\ddot{t}+1]{R'} m'_3$. Since $m_3 \in U'_{\ddot{t}+1}$, $\hat{m}_3 \in U'_{\ddot{t}+1}$ and $\hat{m}_3 \preceq m_3$. Since $m'_2 \in S'_{\ddot{t}+1}$, $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2$, and $\hat{m}_3 \in U'_{\ddot{t}+1}$, we have $m'_3 \in U'_{\ddot{t}+1}$ and $m'_3 \preceq \hat{m}_3$. Thus, $m'_3 \prec k$ which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.

$\boldsymbol{h'_{\ddot{t}+1}(m_2) \neq h_{\ddot{t}+1}(m_2)}$: Let $a \equiv h_{\ddot{t}+1}(m_2)$. By the *induction hypothesis*, since $h'_{\ddot{t}+1}(m_2) \neq a$, we have $m_2 \in S_{\ddot{t}+1}$. Since $j \xrightarrow[\ddot{t}]{R} m_1$, $m_3 \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$.
Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$ and by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t} + 1)$.
Since $a\ I_{m_1}\ h_{\ddot{t}+1}(m_1)$, and $j \not\xrightarrow[\ddot{t}+1]{R'} m_1$, we have that $m_1 \in S'_{\ddot{t}+1}$, $m_1 \xrightarrow[\ddot{t}+1]{R'} S'_{\ddot{t}+1}$, and $\hat{m} \in S'_{\ddot{t}+1}$.
Since $h_{\ddot{t}+1}(\hat{m}) \neq a$ and since there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.
Now consider the first $\check{t}$, which is between $\hat{t}$ and $\ddot{t}+1$, such that $h_{\check{t}}(m_2) = a$. Then, $m_2 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$ which contradicts $m_3 \in U_{\ddot{t}+1} \subseteq U_{\check{t}-1}$.

$\boldsymbol{h'_{\ddot{t}+1}(m_1) \nVdash h_{\ddot{t}+1}(m_1)}$: Let $a \equiv h_{\ddot{t}+1}(m_1)$. Since $h'_{\ddot{t}+1}(m_1) \neq a$, $m_1 \in S_{\ddot{t}+1}$. Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$. Since $j \xrightarrow[\ddot{t}+1]{R'} j_1$, we have that $\hat{m} \in S'_{\ddot{t}+1}$ and $\hat{m} \xrightarrow[\ddot{t}+1]{R'} S'_{\ddot{t}+1}$. Since $h_{\ddot{t}+1}(\hat{m}) \neq a$, by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t} + 1)$ and there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$. Since $\hat{m} \in S'_{\ddot{t}+1}$ and $p'_{\ddot{t}+1}(\hat{m}) = p'_{\hat{t}}(\hat{m})$, we have that $\hat{m} \in S'_{\hat{t}}$. This implies that $\hat{m} \in S_{\hat{t}}$ and $h_{\hat{t}}(\hat{m}) = a$. Since $\hat{m} \xrightarrow[\hat{t}]{R'} S'_{\hat{t}}$, then $\hat{m} \xrightarrow[\hat{t}]{R} S_{\hat{t}}$. And for each $\hat{\hat{t}} > \hat{t}$, we have $\hat{m} \xrightarrow[\hat{\hat{t}}]{R} S_{\hat{\hat{t}}}$. Now consider the first $\check{t}$, which is between $\hat{t}$ and $\ddot{t} + 1$, such that $h_{\check{t}}(m_1) = a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1} \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$. However, if $m_2 \in U_{\ddot{t}+1} \subseteq U_{\check{t}}$, then $m_1 \xrightarrow[\ddot{t}+1]{R} U_{\ddot{t}+1} \subseteq U_{\check{t}}$ and if $m_2 \in S_{\ddot{t}+1}$, then $m_3 \in U_{\ddot{t}+1}$ and $m_1 \xrightarrow[\ddot{t}+1]{R} S_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R} U_{\ddot{t}+1}$. In either case, we have reached a contradiction.

Stage . . .) Repeating this argument for the rest of the pointing phase we show (iii).

Finally, we prove (iv) for Step $\ddot{t}+1$. That is, we show that for each $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$. By (iii) each *trading cycle* that does not involve people connected to i under R' is also a *trading cycle* under R. Therefore, for each $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$. Moreover, for each $j \in CONN(i, R', \ddot{t}+1)$, $h'_{\ddot{t}+2}(j) = h'_{\ddot{t}+1}(j)$. $\diamond$

B Proof of Propositions 1 and 2

Proposition 1: *If $N > 2$, no rule is strategy-proof, Pareto-efficient and anonymous.*

Proof: Let φ by a rule satisfying the axioms. We prove this for the case of $N = 3$.

Suppose φ is a *strategy-proof* and *Pareto-efficient*. Let $O = \{a, b, c\}$, $N = \{1, 2, 3\}$, and let $\omega = (a, b, c)$. Consider the following preference profile:

R_1	R_2	R_3
$a\ b\ c$	a	Ⓐ
	$b\ c$	$b\ c$

By *efficiency*, $\varphi(R, \omega)(1) \neq a$. Thus, either $\varphi(R, \omega)(2) = a$ or $\varphi(R, \omega)(3) = a$. Suppose $\varphi(R, \omega)(3) = a$.

Claim *(Limited favoritism)*: *If 1 is indifferent between all three objects, and if 3's unique most preferred object is a, it is assigned to him. That is, for each $R' \in \mathcal{R}^N$,*[18]

$$\left.\begin{array}{l} R'_1 = \overline{I_0}, \text{ and} \\ \tau(R'_3, O) = \{a\} \end{array}\right\} \Rightarrow \varphi(R', \omega)(3) = a.$$

Proof: By *strategy-proofness*, for each $R'_3 \in \mathcal{R} \setminus \{R_3\}$ such that $\tau(R'_3, O) = \{a\}$, $\varphi(R'_3, R_{-3}, \omega)(3) = a$. Otherwise,

$$\varphi(\underbrace{R_3}_{\text{lie}}, R_{-3}, \omega)(3) \underbrace{P'_3}_{\text{truth}} \varphi(\underbrace{R'_3}_{\text{truth}}, R_{-3}, \omega)(3).$$

Also by *strategy-proofness*, there is no $R'_2 \in \mathcal{R}$, such that $\varphi(R'_2, R_{-2}, \omega)(2) = a$. Otherwise,

$$\varphi(\underbrace{R'_2}_{\text{lie}}, R_{-2}, \omega)(2) \underbrace{P_2}_{\text{truth}} \varphi(\underbrace{R_2}_{\text{truth}}, R_{-2}, \omega)(2).$$

Thus, for any $R' \in \mathcal{R}^N$ such that $R'_1 = \overline{I_0}$ and $\tau(R'_3, O) = \{a\}$, $\varphi(R', \omega)(3) = a.\diamond$

[18] $\overline{I_0}$ is indifference between all objects.

Since φ exhibits *limited favoritism*, it cannot be *anonymous*. ■

Proposition 2: *If $N > 2$, no rule is strategy-proof, Pareto-efficient, individually rational, and non-bossy.*

Proof: Suppose φ is *strategy-proof, Pareto-efficient, individually rational,* and *non-bossy.* We begin by noting that it satisfies *limited favoritism* as in the proof of the previous proposition.

Claim *(General favoritism)*: *If a is not assigned to 1, then 3 finds his assignment to be at least as good as a. That is, for each $R \in \mathcal{R}^N$,*

$$\varphi(R, \omega)(1) \neq a \Rightarrow \varphi(R, \omega)(3) \; R_3 \; a.$$

Proof: Suppose not. Then, there is $R \in \mathcal{R}^N$ such that $\varphi(R, \omega)(1) \neq a$ and $a \; P_3 \;\; \varphi(R, \omega)(3)$. Let $\alpha \equiv \varphi(R, \omega)$. Since $\alpha(1) \neq a$ and $\alpha(3) \neq a$, we have $\alpha(2) = a$.

Case $b \; P_3 \; a$: Since $\alpha(1) \neq a$, by *individual rationality*, there is $x \in \{b, c\}$ such that $x \; R_1 \; a$. Since $b \; P_3 \; a$, $\alpha(3) \neq b$. Thus, by *Pareto-efficiency*, $\alpha(1) = b$. Further, by *Pareto-efficiency*, $b \; P_1 \; c$ and by *individual rationality*, $b \; R_1 \; a$. There are four possible configurations for the preference profile:

R_1	R_2	R_3
ⓑ	ⓐ	b
⋮	⋮	a
		ⓒ

,

R_1	R_2	R_3
ⓑ	ⓐ b	b
⋮	c	a
		ⓒ

,

R_1	R_2	R_3
ⓑ a	ⓐ	b
c	⋮	a
		ⓒ

or

R_1	R_2	R_3
ⓑ a	ⓐ b	b
c	c	a
		ⓒ

.

The circled allocation in each of the above is α. By *strategy-proofness*, if α is chosen at any one of the four configurations, it is chosen at the first. Thus, it is suffices to show that α cannot be chosen for the first configuration.

Consider the following preference profile:

R'_1	R_2	R_3
ⓑ	ⓐ	b
a c	⋮	a
		ⓒ

By *strategy-proofness*, b is assigned to 1. By *non-bossiness*, a is assigned to 2 and c is assigned to 3.

Now, consider another preference profile:

R'_1	R_2	R'_3
ⓑ	ⓐ	a
a c	$\vdots$	b ⓒ

At (R'_1, R_2, R'_3), by *strategy-proofness*, c is assigned to 3 and by *Pareto-efficiency*, b is assigned to 1 and a is assigned to 2. By *strategy-proofness* and *non-bossiness* the allocation is unchanged for the following profile.

R'_1	R'_2	R'_3
ⓑ	ⓐ	a
a c	c	b ⓒ
	b	

Now suppose 1 reports $\overline{I_0}$,

$\bar{R}_1$	R'_2	R'_3
a ⓑ c	a	ⓐ
	ⓒ	b c
	b	

At $(\bar{R}_1, R'_2, R'_3)$, by *limited favoritism*, a is assigned to 3 and by *Pareto-efficiency* c is assigned to 2, leaving b for 1. But by *strategy-proofness*, b is assigned to 1 at (R'_1, R'_2, R'_3). By *non-bossiness*, the circled allocation cannot be chosen.

Case $a\ P_3\ b$: This case is similar. ◇

Now, we show that *general favoritism* is incompatible with *individual rationality* and *Pareto-efficiency*. Consider the following profile.

$\tilde{R}_1$	$\tilde{R}_2$	$\tilde{R}_3$
b	a	a
a	b	b c
c	c	

By *Pareto-efficiency* and *individual rationality*, b is assigned to 1 and a is assigned to 2. This violates *general favoritism*. ■

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