A market approach to fractional matching

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Abstract

We propose a general fractional matching model. Each person has a set of potential partners and consumes a bundle of partnerships with them. A feasible allocation is one where each person consumes the same quantity of a particular partnership as his partner does. Each person's preferences are defined over partnership bundles.

This model has several natural applications: probability distributions over deterministic matchings for marriage problems, school choice, scheduling different workers at various work sites, organizing paired activities among a group, and so on.

For this novel model, we define a *price* based solution. We show that the *core* of each problem is non-empty. We show that our solution selects a subset of the *core*. We also show that if the number of people involved increases—in a way that there is a fixed number of "kinds" of people—the gains from misreporting preferences diminish.

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1 Introduction

We start with a simple application of our model. Consider a group of players at a tennis club. They are all at the club for the day and have preferences over who

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they play and for how long. The problem is to determine who plays whom, and for how long.

The abstract model that we study is one where each person has a set of potential partners. His consumption space is the set of all combinations of potential partners (i.e. the simplex whose dimension is the number of his potential partners). His preferences are convex, continuous, and locally non-satiated, except at the maxima, over this consumption space. A feasible allocation is one where, for each pair i and j, the amount of partnership with j that i consumes is exactly the same as the amount of partnership with i that j consumes and no person is partnered for more than his availability.

Our model is very general. Among others, it includes the following models as special cases:

1. The fractional (heterosexual) marriage model (Rothblum 1992, Roth, Rothblum and Vande Vate 1993, Aldershof, Carducci and Lorenc 1999, Baïou and Balinski 2000, Klaus and Klijn 2006, Bogomolnaia and Moulin 2004, Sethuraman, Teo and Qian 2006, Manjunath 2011): Since the potential partners are determined by a bi-partition, any feasible allocation is a bistochastic matrix and thus a probability distribution over deterministic matchings. In fact, our model covers a fractional version of the *roommate problem* (Gale and Shapley 1962). However, fractional matchings for the roommate problem cannot necessarily be expressed as probability distributions over deterministic matchings (Budish, Che, Kojima and Milgrom 2010).

2. The model of trade under bilateral constraints (Bochet, İlkılıç, Moulin and Sethuraman 2010): This is the case where each person is indifferent between partners and has single peaked preferences over the time that he spends with other people (rather than being alone).

3. School choice (Abdulkadiroğlu and Sönmez 2003, Abdulkadiroğlu, Pathak, Roth and Sönmez 2005, Erdil and Ergin 2008, Abdulkadiroğlu, Che and Yasuda 2010): While a school does not have preferences over the children that are admitted to it, each school is associated with a priority order over children. These priorities dictate which children are to be favored at each school. Just as in the fractional marriage model, feasible allocations are bistochastic matrices.

4. A model of matching workers with different skills to various employers who desire particular combinations of skills. We interpret a feasible allocation as a schedule that determines the time each worker spends at each job. To the extent of our knowledge, though related to the "stable schedule problem" (Baïou and Balinski 2002, Alkan and Gale 2003), the model presented in Section 6.4 is novel.

Our main contributions are:

1. We propose a new model that encompasses all of the above. We show that it can be interpreted as a production model. A key insight is that a partnership is a public good in some senses and a private good in others. It is like a public good in that if a person i consumes a certain amount of partnership with j, then j necessarily consumes the same amount of partnership with i. It is a private good in that i excludes all others from that amount of partnership with j. Understanding this helps us define a price-based solution for such economies. Since "resources have preferences," the price system that we consider will have to reflect them. We achieve this through double-indexed prices. A natural interpretation for "double-indexed" prices is that since each i has preferences over whom he partners with, it is natural that he would charge different partners different prices. Specifically, if he prefers j to k, then he would charge j less than he would charge k.

2. We define an appropriate notion of the *core* and show that it is nonempty for each economy. We do so by proving that the non-transferable-utility (NTU) game associated with each economy is "balanced" (Scarf 1967). We also show that our price-based solution selects a subset of the *core*.

3. We add some structure to our model to incorporate the concept of a "kind" of person. In the more general version of our model, each person has preferences over his potential partners. Think of each person as having a set of "external characteristics" or a *kind*. For the examples listed above, these could be a man or woman's income, and education, a trader's connections, a school child's proximity to a school and number of siblings, and a worker's skills. Not only are these characteristics observable by others, but they are what preferences are based on.

   We consider a generalization of our model where a person is identified by a *kind* along with his preferences over bundles of kinds. We extend our solution by indexing prices by kind rather than identity. This is important for two reasons. First, from a fairness point of view, two people who are exactly the same ought to be given exactly the same opportunities. Second, if the market is "thick" (there are many people of each kind), for given utility representations, the gains from misreporting preferences is small for each person of each kind.

While Shapley and Shubik (1969) have defined *competitive equilibria* for matching problems, we remind the reader that the model that they consider involves monetary transfers. Subsequently, Kelso and Crawford (1982) have shown the nonemptiness of the *core* and its equivalence with the set of *competitive allocations* for matching markets involving money. Bikhchandani and Ostroy (2002) and Sun and Yang (2006) have also used non-anonymous prices. Our model differs from the ones studied in these papers in two important ways: 1) the goods in our model are divisible and 2) monetary transfers are not possible in our model.

Allocation models where resources are not associated with preferences can be encoded as instances of the model that we study. However, the analysis presented here is not as interesting as it is for the case where resources *are* associated with preferences. In particular, we need not resort to double-indexed prices since single-indexed *competitive equilibria from equal income* typically do exist for these problems (Hylland and Zeckhauser 1979, Budish 2010).

Though there are papers on two-sided "probabilistic" (or fractional) matching, such as those mentioned above, their focus has been on the *ex-post core*. Exceptions are Bogomolnaia and Moulin (2004) and Manjunath (2011). However, Bogomolnaia and Moulin (2004) study problems where preferences are "dichotomous." For this very restricted class of problems, they propose a rule that fulfills certain efficiency and fairness criteria. Manjunath (2011) studies various *ex-ante core* notions and their logical relations.

The remainder of the paper is organized as follows. In Section 2 we formally introduce the model and define key concepts. In Section 4 we prove that the *core* is never empty. In Section 5 we prove that our solution is well defined. We particularize our model for specific applications in Section 5. In Section 7 we generalize our model to accommodate *kinds* of people.

2 The Model

Let N be a set of people. Each $i \in N$ is associated with a set of **potential partners** $\boldsymbol{S_i} \subseteq N$ such that $i \in S_i$.[1] For each $i \in N$, i's **consumption set** is $\Delta(S_i)$. Let $\boldsymbol{R_i}$ be i's preference relation over $\Delta(S_i)$. We require that R_i be continuous, convex, and locally non-satiated except at its maxima on $\Delta(S_i)$. Let $\mathcal{R}_i$ be the set of all such preferences. For each pair $x, y \in \Delta(S_i)$, if i finds x to be at least as desirable as y under preference relation R_i, we write $x \; R_i \; y$. Similarly, if i prefers x to y, we write $x \; P_i \; y$. If he is indifferent between them, we write $x \; I_i \; y$.

An **economy** is described by a profile of preferences $R \in \mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$.

A **feasible allocation** specifies for each $i \in N$ a consumption bundle $\pi_i \in \Delta(S_i)$ in a way that for each $i \in N$ and each $j \in S_i$, $\pi_{ij} = \pi_{ji}$. Let $\boldsymbol{\Pi}$ be the set of feasible

[1] While this specification of *potential partners* might remind the reader of that in Sönmez (1996), it ought to be noted that Sönmez's analysis is not restricted to bilateral situations such as ours.

allocations. We represent a feasible allocation by a symmetric $N \times N$ matrix, the rows and columns of which sum to one.

A **solution**, $\phi : \mathcal{R} \rightrightarrows \Pi$, associates each economy with a set of feasible allocations.

3 Solutions

We start with some normatively appealing solutions. The first one reflects a very familiar notion of *efficiency*. For each $R \in \mathcal{R}$ and $\pi \in \Pi$, we say that π is **Pareto-efficient at $\boldsymbol{R}$** if there is no $\pi' \in \Pi$ such that for each $i \in N, \pi'_i \; P_i \; \pi_i$. Let $\boldsymbol{P(R)}$ be the set of *Pareto-efficient allocations at R*.

The next solution expresses the principle that each person has the right to "consume" himself. Let $\boldsymbol{\delta} \in \Pi$ be such that for each $i \in N$, $\delta_{ii} = 1$. For each $R \in \mathcal{R}$ and $\pi \in \Pi$, we say that π is **individually rational at $\boldsymbol{R}$** if for each $i \in N, \pi_i \; R_i \; \delta_i$. Let $\boldsymbol{I(R)}$ be the set of allocations that are *individually rational at R*.

At one end, we have defined the *Pareto* solution that picks allocations that society as a whole cannot improve upon. At the other end, the *individual rationality* solution respects the rights of individuals. The following solution extends these principles to groups of all sizes. For each $R \in \mathcal{R}$, $\pi \in \Pi$, and $S \subseteq N$, **$\boldsymbol{S}$ blocks $\boldsymbol{\pi}$ at $\boldsymbol{R}$** if there is $\pi^S \in \Pi$ such that for each $i \in S$,

$$i) \quad \sum_{j \in S} \pi^S_{ij} = 1 \text{ and}$$

$$ii) \quad \pi^S_i \; P_i \; \pi_i.$$

The **core at $\boldsymbol{R}$, $\boldsymbol{C(R)}$,** is the set of allocations that are not *blocked* by any coalition at R. We will show that $C(R)$ is never empty.

We begin our search for a price-based solution with a naïve first attempt. Since each person "owns" himself, we assign a price to each person and allow people to trade parts of themselves for parts of others. An allocation $\pi \in \Pi$ is a **Walrasian allocation at** R if there is a vector $p \in \mathbb{R}^N$ such that for each $i \in N$,

$$\begin{gathered} \pi_i \in \underset{\pi'_i \in \Delta(S_i)}{\operatorname{argmax}} \; R_i \\ \text{subject to} \\ \underbrace{\sum_{j \in S_i} \pi_{ij} p_j}_{\text{Price of } \pi'_i} \leq \underbrace{1 \cdot p_i}_{i\text{'s income}} . \end{gathered}$$

We refer to (π, p) as a **Walrasian equilibrium**. Let $\boldsymbol{W(R)}$ be the set of all *Walrasian allocations at R*. As demonstrated by Example 1, $W(R)$ may be empty.

Example 1. *An economy with no* Walrasian allocation.

Let $N \equiv \{m_1, m_2, w_1, w_2\}$ and

$$\begin{aligned} S_{m_1} &\equiv \{m_1, w_1, w_2\}, \\ S_{m_2} &\equiv \{m_2, w_1, w_2\}, \\ S_{w_1} &\equiv \{w_1, m_1, m_2\}, \text{ and} \\ S_{w_1} &\equiv \{w_2, m_1, m_2\}. \end{aligned}$$

Let $R \in \mathcal{R}$ be such that the following are numerical representations:

$$\begin{aligned} &\text{For each } \pi_{m_1} \in \Delta(S_{m_1}), u_{m_1}(\pi_{m_1}) = 2\pi_{m_1 w_1} + \pi_{m_1 w_2}, \\ &\text{for each } \pi_{m_2} \in \Delta(S_{m_2}), u_{m_2}(\pi_{m_2}) = 2\pi_{m_2 w_2} + \pi_{m_2 w_1}, \\ &\text{for each } \pi_{w_1} \in \Delta(S_{w_1}), u_{w_1}(\pi_{w_1}) = 2\pi_{w_1 m_2} + \pi_{w_1 m_1}, \text{ and} \\ &\text{for each } \pi_{w_2} \in \Delta(S_{w_2}), u_{w_2}(\pi_{w_2}) = 2\pi_{w_2 m_1} + \pi_{w_2 m_2}. \end{aligned}$$

Let $p \in \mathbb{R}^N_+$. Suppose that (π, p) is a *Walrasian equilibrium*. Suppose $m_1 \in \underset{i \in N}{\text{argmax}}\, p_i$. Then, $\pi_{m_1 w_1} = 1$. By feasibility, $\pi_{m_2 w_1} = 0$. So, $p_{w_1} > p_{m_2}$. This implies that $\pi_{w_1 m_2} = 1$ and contradicts $\pi_{m_1 w_1} = 1$. Since the problem is symmetric (each person has the same preferences over bundles of being single, with the most preferred mate and with the least preferred mate), we reach a similar contradiction if $m_1 \notin \underset{i \in N}{\text{argmax}}\, p_i$. ◦

The reason that a *Walrasian allocation* may not exist is that some of the consumption goods in these economies are *not* private goods: a partnership involves both members.

For our next attempt to define a price-based solution, we draw inspiration from the literature on public goods economies and introduce "double-indexed" prices, as follows.

Let $M \subseteq N \times N$ be such that $(i, j) \in M$ if and only if $j \in S_i$ and $i \in S_j$. We say that $\pi \in \times_{i \in N} \Delta(S_i)$ is a **double-indexed price (DIP) allocation at** R if there is a vector $p \in \mathbb{R}^M_+$ such that for each $i \in N$,

$$\begin{gathered} \pi_i \in \underset{\pi'_i \in \Delta(S_i)}{\text{argmax}}\, R_i \\ \text{subject to} \\ \underbrace{\sum_{j \in S_i} \pi'_{ij} p_{ij}}_{\text{Price of } \pi'_i} \leq \underbrace{\sum_{j \in S_i} \pi_{ji} p_{ji}}_{i\text{'s income at } \pi}, \end{gathered}$$

and $\pi \in \Pi$ (this ensures that the "market clears"). We interpret the price vector as follows: for each $(i, j) \in M$, p_{ij} is the price that i pays for j.

We refer to (π, p) as a **double-indexed price equilibrium**. Let $\boldsymbol{D(R)}$ be the set of all *DIP allocations at R*.

It is easy to see that a *Walrasian equilibrium*, if it exists, is also a *DIP equilibrium*: Let (π, q) be a *Walrasian equilibrium* and define $p \in \mathbb{R}^M_+$ by setting, for each $i \in N$ and each $j \in S_i, p_{ji} = q_i$. It follows directly from the two definitions that (π, p) is a *DIP equilibrium*.

Remark 1. Our definition of a *DIP equilibrium* has the flavor of a "Lindahl equilibrium" (Lindahl 1958). The reason is that a positive amount of a partnership between i and $j \in N$ is not a private good (nor is it a pure public good). If i consumes a certain amount of this partnership, say π_{ij}, then he excludes all others from consuming it. Yet, j is not excluded. Note that partnerships are not "common goods" or "club goods" either.[2] △

Unfortunately, even *DIP allocations* may not exist as demonstrated by Example 2.

Example 2. *An economy with no* DIP allocation.

Let $N \equiv \{1, 2\}$ and for each $i \in N$, $S_i \equiv N$. Let $R \in \mathcal{R}$ be such that for each $i \in N, R_i$ is represented by $u_i : \Delta(S_i) \to \mathbb{R}$ defined as follows:

$$\begin{aligned}&\text{For each } \pi_1 \in \Delta(S_1), u_1(\pi_1) = \pi_{12} \text{ and}\\ &\text{for each } \pi_2 \in \Delta(S_2), u_2(\pi_2) = -(\tfrac{1}{4} - \pi_{21})^2.\end{aligned}$$

Suppose that $(\pi, p) \in \Pi \times \mathbb{R}^M_+$ is a *DIP equilibrium*. Only the relationship between the prices p_{12} and p_{21} is relevant. For each possibility, we show that $\pi \notin \Pi$: If $p_{12} > p_{21}$, then $\pi_{12} = 0$ and $\pi_{21} = 1$. If $p_{12} < p_{21}$, then $\pi_{12} = \frac{1}{4}$ and $\pi_{21} = 0$. Finally, if $p_{12} = p_{21}$, then $\pi_{12} = \frac{1}{4}$ and $\pi_{21} = 1$. ○

The difficulty here arises from the fact that "endowments" of each person are on the boundaries of their consumption spaces. To deal with this, we redistribute a small amount of each person's endowment. Before proceeding, we define some useful notation: for each $S \subseteq N$, let $\Lambda(S) = \{\lambda \in \mathbb{R}^S_+ : \sum_{i \in S} \lambda_i \leq 1\}$. Clearly, for each $i \in N, \Delta(S_i) \subset \Lambda(S_i)$. For each $R \in \mathcal{R}$, and each $i \in N$, let $\hat{R}_i$ be an extension of R_i from $\Delta(S_i)$ to $\Lambda(S_i)$ such that $\hat{R}_i$ is:

- *strictly monotonic* over $\Lambda(S_i) \setminus \Delta(S_i)$,

[2] A *common good* is one that is not excludable but has congestion effects. A *club good* is one that can be consumed by any number of people simultaneously but is excludable.

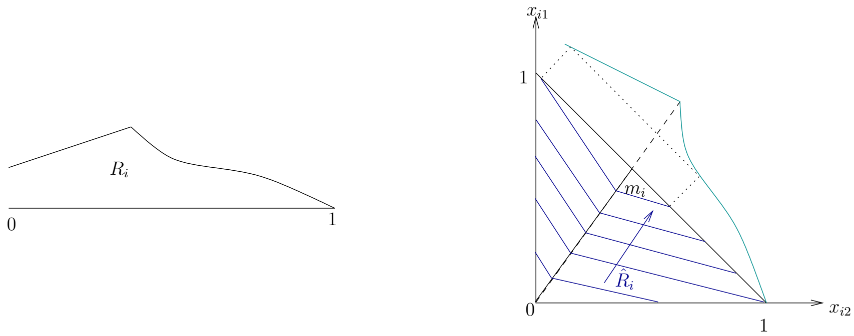

Figure 1: The preference relation $\hat{R}_i$ which is an extension of R_i from $\Delta(S_i)$ to $\Lambda(S_i)$.

- *continuous*, and
- *convex*.

Claim 1. *Such $\hat{R}$ exists.*

Proof: We describe the construction of one such profile (see Figure 1). First we extend R_i to $\{x_i \in \mathbb{R}^{S_i} : \sum_{j \in S_i} x_{ij} = 1\}$ and then define $\hat{R}_i$ over $\Lambda(S_i)$.

Let $\tilde{R}_i$ be a continuous and convex extension of R_i from $\Delta(S_i)$ to $\{x_i \in \mathbb{R}^{S_i} : \sum_{j \in S_i} x_{ij} = 1\}$ that is locally non-satiated except at the maxima of R_i over $\Delta(S_i)$.

Let $M_i \equiv \underset{\Delta(S_i)}{\text{argmax}}\ R_i$. For each $x \in \Delta(S_i)$, let $I_i(x) \equiv \{y \in \Delta_i : x_i\ I_i\ y_i\}$. For each $x \in \Delta(S_i)$, let

$$d(x) = \min_{\substack{y \in M_i \\ z \in I_i(x)}} ||z - y||.$$

That is, $d(x)$ is the shortest distance between the indifference class of x and M_i. Note that for all x, $d(x) \leq \sqrt{2}$. Define $\hat{R}_i$ as follows: for each $x_i \in \Delta(S_i)$, let the upper contour set of $\hat{R}_i$ at x_i be

$$U(\hat{R}_i, x) \equiv \text{convex hull}\left(\left(1 - \frac{d(x)}{2}\right) M_i \cup U(\tilde{R}_i, x_i)\right) \cap \mathbb{R}_+^{S_i}.$$

Let $w_i \in \{x_i \in \Delta(S_i) : \text{for each } y_i \in \Delta(S_i), y_i\ R_i\ x_i\}$. That is, w_i is one of i's least preferred points in $\Delta(S_i)$ at preference relation R_i. The preference map over $\Lambda(S_i) \setminus U(\hat{R}_i, w_i)$ is completed by translating the level set of w_i.

We verify the following facts about $\hat{R}_i$:

- **Convexity:** By definition of $\hat{R}_i$, at each $x_i \in \Lambda(S_i), U(\hat{R}_i, x_i)$ is convex.

- **Continuity:** By definition of $\hat{R}_i$, at each $x_i \in \Lambda(S_i)$, $U(\hat{R}_i, x_i)$ is closed.

- **Strict monotonicity:** We check that it is also *strictly monotone* over $\Lambda(S_i) \setminus \Delta(S_i)$. Let $x_i \in \Lambda(S_i) \setminus \Delta(S_i)$. Then, there are $y_i \in \Delta(S_i)$, $m_i \in M_i$, and $\alpha \in [0, 1)$ such that

$$x_i = \alpha y_i + (1 - \alpha) \left(1 - \frac{d(y_i)}{2}\right) m_i.$$

  Let $z_i \equiv \left(1 - \frac{d(y_i)}{2}\right) m_i$. Consider the triangle formed by the vectors m_i, z_i, and y_i. It suffices to show that the angle at the vertex y_i is less than 45°. Since $U(\hat{R}_i, x_i)$ is comprehensive and $x_i \; \hat{I}_i \; y_i$, the angle at y_i is no more than 45°. Suppose it is 45°. Since the distance between y_i and m_i is at least $d(y_i)$, the distance between m_i and z_i is at least $\frac{d(y_i)}{\sqrt{2}}$. However, this is a contradiction since $||z_i - m_i|| = \frac{d(y_i)}{2} < \frac{d(y_i)}{\sqrt{2}}$. Finally, since $\tilde{R}_i$ is locally non-satitated except at M_i, we conclude that $\hat{R}_i$ is strictly monotonic over $\Lambda(S_i) \setminus \Delta(S_i)$. □

We are now ready to define our next solution.

Let $\varepsilon \in (0, 1)$. An allocation $\pi \in \Pi$ is an **ε-double-indexed price (εDIP) allocation** if there is $p \in \mathbb{R}^M_+$ such that:

1. For each pair $(i, j) \in M$,

$$p_{ii} + p_{jj} \geq p_{ij} + p_{ji}.$$

2. For each pair $(i, j) \in M$ such that $\pi_{ij} > 0$.

$$p_{ii} + p_{jj} = p_{ij} + p_{ji}.$$

3. For each $i \in N$,

$$\pi_i \in \underset{x_i \in \Lambda(S_i)}{\text{argmax}} \; R_i$$
$$\text{subject to}$$
$$\underbrace{\sum_{j \in S_i} x_{ij} p_{ij}}_{\text{Price of } x_i'} \leq \underbrace{(1 - \varepsilon) p_{ii} + \left(\frac{\varepsilon}{|n| - 1}\right) \left(\sum_{j \in N \setminus \{i\}} p_{jj}\right)}_{i\text{'s income}}.$$

We refer to (π, p) as an **εDIP equilibrium**.

The two conditions on the price vector are profit maximization conditions for hypothetical firms that produce partnerships, taking the individuals as inputs. Let $\boldsymbol{D^{\varepsilon}(R)}$ be the set of all *εDIP allocations at R*. As we will show, for each $\varepsilon \in (0,1)$ and each $R \in \mathcal{R}$, there is an *εDIP* allocation.

Note that if we set $\varepsilon = 0$, the above definition coincides with that of a *DIP* allocation.

We now define our last solution. An allocation $\pi \in \Pi$ is a **limit DIP (lim-DIP) allocation** if there is a sequence $\{\pi^{\varepsilon}\}_{\varepsilon \in (0,1)} \in \Pi$ such that

$$\begin{aligned} i) &\quad \text{for each } \varepsilon \in (0,1), \pi^{\varepsilon} \in D^{\varepsilon}(R) \text{ and} \\ ii) &\quad \lim_{\varepsilon \to 0} \pi^{\varepsilon} = \pi. \end{aligned}$$

Let $\boldsymbol{D^{l}(R)}$ be the set of all *limit DIP allocations*. In Section 5 We will show that $D^{l}(R) \subseteq C(R)$.

4 Existence of a core allocation

Theorem 1. *For each $R \in \mathcal{R}$, $C(R) \neq \varnothing$.*

Proof: Let $R \in \mathcal{R}$. We proceed by associating with R an NTU game and then showing that it is "balanced."

For each $i \in N$, let $u_i : \Delta(S_i) \to \mathbb{R}$ be a numerical representation of R_i. That is, for each pair $\pi_i, \pi_i' \in \Delta(S_i)$,

$$u_i(\pi_i) \geq u_i(\pi_i') \Leftrightarrow \pi_i \; R_i \; \pi_i'.$$

For each $S \subseteq N$, let

$$V^S \equiv \left\{ v_S \in \mathbb{R}^S : \text{ there is } \pi^S \in \Pi \text{ such that for each } i \in S, \begin{array}{c} \sum_{j \in S} \pi_{ij}^S = 1 \text{ and} \\ v_i \leq u_i(\pi_i^S) \end{array} \right\}.$$

Let $T \subseteq \mathbb{P}(N)$ be a collection of subsets of N. If there is $(\delta_S)_{S \in T} \in \mathbb{R}_+^T$ such that for each $i \in N$,

$$\sum_{\substack{S \in T \\ \text{s.t. } i \in S}} \delta_S = 1,$$

then T is **balanced**.

The game $(V^S)_{S \in N}$ is **balanced** if for each $v \in \mathbb{R}^N$ and each *balanced collection T*,

if for each $S \in T, v_S \in V^S$, then $v \in V^N$.[3]

By verifying that $(V^S)_{S\subseteq N}$ is *balanced*, we conclude that $C(R) \neq \varnothing$ (Scarf 1967).[4] Let T be a *balanced collection* with weights δ. Let $v \in \mathbb{R}^N$ be such that

$$\text{for each } S \in T, v_S \in V^S.$$

Then, for each $S \in T$, there is $\pi^S \in \Pi$ such that for each $i \in S$, $\sum_{j\in S} \pi^S_{ij} = 1$ and $v_i \leq u_i(\pi^S_i)$. Define $\pi \in \mathbb{R}^{N\times N}_+$ by setting for each $i \in N$ and each $j \in S_i$,

$$\pi_{ij} \equiv \begin{cases} \sum_{\substack{S\in T \\ \text{s.t. } i\in S}} \delta_S \pi^S_{ij} & \text{if } j \in S_i \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For each $i \in N$,

$$\sum_{j\in S_i} \pi_{ij} = \sum_{j\in S_i} \sum_{\substack{S\in T \\ \text{s.t. } i\in S}} \delta_S \pi^S_{ij} = \sum_{\substack{S\in T \\ \text{s.t. } i\in S}} \delta_S \sum_{j\in S_i} \pi^S_{ij} = \sum_{\substack{S\in T \\ \text{s.t. } i\in S}} \delta_S \cdot 1 = 1.$$

Further, for each $j \in S_i \setminus \{i\}$ and each $S \in T$, $\pi^S_{ij} = \pi^S_{ji}$ and,

$$\pi_{ij} = \sum_{\substack{S\in T \\ \text{s.t. } i\in S}} \delta_S \pi^S_{ij} = \sum_{\substack{S\in T \\ \text{s.t. } i,j\in S}} \delta_S \pi^S_{ij} = \sum_{\substack{S\in T \\ \text{s.t. } j\in S}} \delta_S \pi^S_{ji} = \pi_{ji}.$$

Thus, $\pi \in \Pi$.

For each $i \in N$, since π_i is a convex combination of $(\pi^S_i)_{\substack{S\in T \\ \text{s.t. } i\in S}}$ and since R_i is convex, $u_i(\pi_i) \geq \min_{\substack{S\in T \\ \text{s.t. } i\in S}} u_i(\pi^S_i) \geq v_i$ and $v \in V^N$. Thus $(V^S)_{S\subseteq N}$ is balanced and $C(R) \neq \varnothing$. □

5 Nonemptiness of *lim-DIP*

We first show that an *εDIP* always exists. Since Π is compact, we then conclude that a *lim-DIP* exists (Theorem 2).

Proposition 1. *For each $\varepsilon \in (0,1)$ and $R \in \mathcal{R}$, $D^\varepsilon(R) \neq \varnothing$.*

[3] For each $v \in \mathbb{R}^N$ and each $S \subseteq N$, we denote the projection of v onto the coordinates in S by v_S.

[4] Note that we appeal to the sufficient condition for nonemptiness of the core for NTU games by Scarf (1967) rather than the necessary and sufficient conditions for TU games by Bondareva (1962) and Shapley (1965).

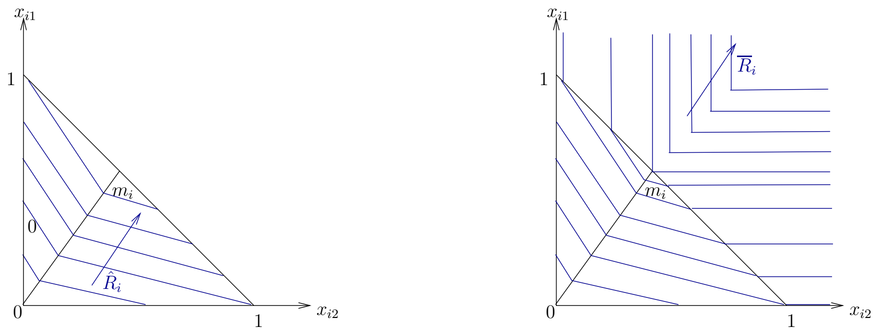

Figure 2: The preference relation $\hat{R}_i$ which is an extension of R_i from $\Delta(S_i)$ to $\Lambda(S_i)$.

Proof: We proceed by embedding R in an Arrow-Debreu model. We then show the existence of a *competitive equilibrium* of this augmented economy (McKenzie 1959, Arrow and Hahn 1971). We conclude by showing that this *competitive equilibrium* corresponds to an εDIP of R.

Step 1: *Embed R in a classical economy.*

For each $i \in N$, let i's consumption space be $X_i \subseteq \mathbb{R}^M_+$ defined by

$$x \in X_i \Leftrightarrow \ (x_{ij})_{j \in S_i} \in \mathbb{R}^{S_i}_+ \text{ and for each pair } (j,k) \in M \text{ such that } j \neq i, x_{jk} = 0.$$

That is, $X_i \equiv \mathbb{R}^{S_i}_+ \times \{(0,\ldots,0)\}$. By definition, X_i is closed and convex.

A notable feature of these consumption spaces is that for each pair $i, j \in N$ $X_i \cap X_j = \{0\}$.

Note that $\Lambda(S_i) \times \{0\} \subseteq X_i$. For each $i \in N$, we now extend $\hat{R}_i$ (defined in Section 3) from $\Lambda(S_i)$ to $\overline{R}_i$ over X_i (see Figure 2).

For each $x_i \in \Lambda(S_i)$, let the upper contour set of $\overline{R}_i$ at x_i be

$$U(\overline{R}_i, x_i) \equiv \text{comp } U(\hat{R}_i, x_i).$$[5]

Since for each $m_i \in M_i$, $U(\overline{R}_i, m_i)$ is convex, the preference map in this region is completed by translating the level set of M_i.

Clearly, $\overline{R}_i$ is *continuous*, *convex*, *monotone*, and *strictly monotone over* $\Lambda(S_i) \setminus \Delta(S_i)$

Let F be a set of $\frac{|M|-|N|}{2}$ firms. Label these firms by unordered (distinct) pairs from N, a generic member being $\{i,j\}$. The production set of $\{i,j\} \in F$ is

$$Y_{\{i,j\}} \equiv \left\{ y \in \mathbb{R}^{\{ij,ji,ii,jj\}} : y_{ij} = y_{ji} = -y_{ii} = -y_{jj} \right\} \times \{0\} \subset \mathbb{R}^M.$$

[5]Denote the "upper comprehensive hull" of $X \subseteq \mathbb{R}^l$ by comp $(X) \equiv X + \mathbb{R}^l_+$.

Note that $Y_{\{i,j\}}$ is closed and convex.

For each $\{j,k\} \in F$ and each $i \in N$, let $\sigma_i(\{j,k\})$ be i's share of $\{j,k\}$. Thus, for each $\{j,k\} \in F, \sum_{i \in N} \sigma_i(\{j,k\}) = 1$.

Finally, for each $i \in N$, let $\omega^i \in \mathbb{R}^M_+$ be such that for each pair $k, j \in N$,

$$\omega^i_{kj} = \begin{cases} 1-\varepsilon & \text{if } i=j=k, \\ \frac{\varepsilon}{|N|-1} & \text{if } i \neq j = k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\omega \equiv (\omega^i)_{i \in N}$.

We have now specified an exchange economy with production $E \equiv (X, Y, \overline{R}, \omega, \sigma)$.

Step 2: *Check that E has a competitive allocation.*

Since the set of goods that each person is endowed with is the same, E is "irreducible" (McKenzie 1959) (alternatively, we could have shown that it satisfies "resource relatedness" (Arrow and Hahn 1971)). Let $Y \equiv \sum_{\{i,j\} \in M} Y_{\{i,j\}} + \omega$ and $X \equiv \sum_{i \in N} X_i$. We have the following:

1. For each $i \in N$, X_i is convex, closed, and bounded from below.
2. For each $i \in N$, $\overline{R}_i$ is continuous, convex, and weakly monotonic.
3. For each $i \in N, X_i \cap Y \neq \varnothing$.
4. For each $\{i,j\} \in M$, $Y_{\{i,j\}}$ is closed and convex.
5. $Y \cap \mathbb{R}^M_+ = \{0\}$.
6. ω is in the relative interiors of Y and X.
7. Irreducibility (McKenzie 1959): For each bi-partition N_1, N_2 of N, if $x_{N_1} \in Y - \sum_{i \in N_2} X_i$, then there is $w \in Y - \sum_{i \in N_2} X_i$ and $x' \in X$ such that $w = \sum_{i \in N_1} x'_i - \sum_{i \in N_2} x_i$ and for each $i \in N_1, x'_i \; R_i \; x_i$ with $x'_i \; P_i \; x_i$ for at least one $i \in N_1$.

By Theorem 2 of McKenzie (1959), E has a *competitive allocation* $(x, y, p) \in X \times Y \times \mathbb{R}^M_+$.

Step 3: *Show that $(x,p) \in D^\varepsilon(R)$.*

We check that for each $i \in N, x_i \in \Delta(S_i)$. From this, we conclude that $x \in D^\varepsilon(R)$. Suppose that there is $i \in N$ such that $x_i \notin \Delta(S_i)$.

Case 1: $\sum_{j\in S_i} x_{ij} > 1$. Then, $\sum_{j\in S_i\setminus\{i\}} x_{ij} + x_{ii} > 1 = \sum_{j\in N} \omega^j_{ii}$.

For each $j \in S_i \setminus \{i\}$, $x_{ij} \leq y^{\{i,j\}}_{ij} = -y^{\{i,j\}}_{ii}$.

Thus, $\sum_{j\in S_i\setminus\{i\}} x_{ij} \leq -\sum_{j\in S_i\setminus\{i\}} y^{\{i,j\}}_{ii}$,

Finally, we establish that $x_{ii} \geq \omega_{ii} + \sum_{j\in S_i\setminus\{i\}} y^{\{i,j\}}_{ii}$. This violates the feasibility of (x, y) for E.

Case 2: $\sum_{j\in S_i} x_{ij} < 1$. Let $\alpha = 1 + \sum_{j\in S_i\setminus\{i\}} y^{\{i,j\}}_{ii}$. By feasibility, $\alpha \geq x_{ii}$. If $\alpha > x_{ii}$ then, let $x' \in X$ be such that $x'_{ii} = \alpha$ and for each $j \in N \setminus \{i\}, x_j{}' = x_j$ and $x'_{ij} = x_{ij}$. Since $\overline{R}_i$ is strictly monotone at x, we know that $x'_i \; \overline{P}_i \; x_i$. This violates the *Pareto-efficiency* of (x, y) at $\overline{R}$ (which is a *competitive allocation* for E).

Thus, $1 + \sum_{j\in S_i\setminus\{i\}} y^{\{i,j\}}_{ii} = x_{ii}$. So, $x_{ii} - \sum_{j\in S_i\setminus\{i\}} y^{\{i,j\}}_{ii} = 1 > \sum_{j\in S_i} x_{ij}$. From this, we conclude that there is $j \in S_i \setminus \{i\}$ such that $x_{ij} < -y^{\{i,j\}}_{ii} = y^{\{i,j\}}_{ij}$. Let $x' \in X$ be such that for each $k \in N \setminus \{i\}, x_k{}' = x_k$, for each $k \in S_i \setminus \{j\}, x'_{ik} = x_{ik}$, and $x'_{ij} = y^{\{i,j\}}_{ij}$. Since $\overline{R}_i$ is strictly monotone at x, we know that $x'_i \; \overline{P}_i \; x_i$. This violates the *Pareto-efficiency* of (x, y) at $\overline{R}$ (which is a *competitive allocation* for E).

Since $\sum_{j\in S_i} x_{ij} = 1$ and $x_{ii} = 1 - \sum_{j\in S_i} y^{\{i,j\}}_{ij}$, we have $\sum_{j\in S_i} x_{ij} = \sum_{j\in S_i} y^{\{i,j\}}_{ij}$. Since for each $j \in S_i \setminus \{i\}, x_{ij} \leq y^{\{i,j\}}_{ij}$, we have $x_{ij} = y^{\{i,j\}}_{ij}$. Since for each $\{i, j\} \in F, y^{\{i,j\}}_{ij} = y^{\{i,j\}}_{ji}$, we deduce that $x_{ij} = x_{ji}$.

Since (x, y, p) is an equilibrium, for each pair $i, j \in N$, if $y^{\{i,j\}}_{ij} = y^{\{i,j\}}_{ij} > 0$ then $p_{ij} + p_{ji} = p_{ii} + p_{jj}$. Otherwise, $p_{ij} + p_{ji} \geq p_{ii} + p_{jj}$.

As we have established, for each $i \in N, x_i \in \Delta(S_i)$. From the definition of $Y_{\{i,j\}}$ for each $\{i, j\} \in M$, we have $x_{ij} = x_{ji}$. Thus, $x \in \Pi$. It is clear that that for each $i \in N$,

$$\left\{\pi_i \in \Delta(S_i) : \sum_{j\in S_i} \pi_{ij}p_{ij} \leq (1-\varepsilon)p_{ii} + \left(\frac{\varepsilon}{|n|-1}\right)\left(\sum_{j\in N\setminus\{i\}} p_{jj}\right)\right\}$$

$$\cap\!\!|$$

$$\left\{x_i \in X_i : \sum_{j\in S_i} x_{ij}p_{ij} \leq p \cdot \omega^i\right\}$$

Thus, (x, p) is a ε*DIP equilibrium at* R and $x \in D^\varepsilon(R)$. □

We now establish that an *lim-DIP* allocation always exists.

Theorem 2. *For each $R \in \mathcal{R}$, $D^l(R) \neq \varnothing$.*

Proof: For each $\varepsilon \in (0,1)$, let $\pi^\epsilon \in D^\varepsilon(R)$ (this is possible since $D^\varepsilon(R) \neq \varnothing$). Since Π is compact, $\pi \equiv \lim\limits_{\varepsilon \to 0} \pi^\varepsilon$ is well defined and $\pi \in D^l(R)$. □

An appealing property of *lim-DIP* is that it is a subsolution of the *core*.

To prove this, we will use the following definitions. Recall the definition, for each $R \in \mathcal{R}$, of $\hat{R}$ in Section 3.

Let $\varepsilon \in (0,1)$. Let $S \subseteq N$ and $\alpha^S \equiv 1 - \left(\frac{|N|-|S|}{|N|-1}\right)\varepsilon$. We say that S **ε-blocks** $\pi \in \Pi$ if there is $x^S \in \times_{i \in S} \Lambda(S_i)$ such that:

1. For each $i \in S$ and each $j \in S \cap S_i$, $x^S_{ij} = x^S_{ji}$, and
2. For each $i \in S$,

$$i) \quad \sum_{j \in S_i \cap S} x^S_{ij} = 1 - \left(\frac{|N|-|S|}{|N|-1}\right)\varepsilon,$$

$$ii) \quad \sum_{j \in S_i \setminus S} x^S_{ij} = 0, \text{ and}$$

$$ii) \quad x^S_i \; \hat{P}_i \; \pi_i.$$

The **ε-core, $\boldsymbol{C^\varepsilon(R)}$**, is the set of allocations are not ε-blocked by any coalition.

Lemma 1. *For each $R \in \mathcal{R}$, $D^\varepsilon(R) \subseteq C^\varepsilon(R)$.*

Proof: Let $\pi \in D^\varepsilon(R)$ and (π, p) is an *εDIP equilibrium*. Suppose that $S \subseteq N$ *ε-blocks* x^S. Then, for each $i \in S$,

$$\sum_{j \in S_i \cap S} p_{ij} x^S_{ij} > p_{ii}(1-\varepsilon) + \left(\sum_{j \in N \setminus \{i\}} p_{jj}\right) \frac{\varepsilon}{|N|-1}.$$

Summing over all members of S,

$$\sum_{i,j \in S} p_{ij} x^S_{ij} > \left(\sum_{i \in S} p_{ii}\right)\left(1 - \frac{|N|-|S|}{|N|-1}\varepsilon\right) + \left(\sum_{i \in N \setminus S} p_{ii}\right) \frac{|S|\varepsilon}{|N|-1}.$$

However, for each $(i,j) \in M$, $p_{ii} + p_{jj} \geq p_{ij} + p_{ji}$ and so,

$$\sum_{i,j \in S} p_{ij} x^S_{ij} \leq \left(\sum_{i \in S} p_{ii}\right)\left(1 - \frac{|N|-|S|}{|N|-1}\varepsilon\right).$$

From this contradiction we conclude that $\pi \in C^{\varepsilon}(R)$. $\square$

Next, we show that the limit of a sequence of *ε-core* allocations, as ε goes to zero, is a *core* allocation.

Lemma 2. *For each $R \in \mathcal{R}$, and each sequence $\{\pi^{\varepsilon}\}_{\varepsilon \in (0,1)}$ such that for each $\varepsilon \in (0,1)$, $\pi^{\varepsilon} \in C^{\varepsilon}(R), \lim_{\varepsilon \to 0} \pi^{\varepsilon} \in C(R)$.*

Proof: Let $\pi \equiv \lim_{\varepsilon \to 0} \pi^{\varepsilon}$. Suppose that $\pi \notin C(R)$. Then there is $S \subseteq N$ and π^S such that for each $i \in S$,

$$\begin{array}{rl} i) & \sum_{j \in S} \pi^S_{ij} = 1 \text{ and} \\ ii) & \pi^S_i \; P_i \; \pi_i. \end{array}$$

Let V be a neighborhood of π and V^S be a neighborhood of π^S such that for each $v \in V$, each $v^S \in V^S$, and each $i \in S$,

$$v^S_i \; \hat{R}_i \; v_i.$$

Since $\hat{R}_i$ is continuous, such V and V^S exist. For ε small enough, $\pi^{\varepsilon} \in V$ and $(1 - \left(\frac{|N|-|S|}{|N|-1}\right) \varepsilon)\pi^S \in V^S$. This contradicts $\pi^{\varepsilon} \in C^{\varepsilon}(R)$. $\square$

We finally establish that the set of *lim-DIP* allocations is a subset of the *core*.

Theorem 3. *For each $R \in \mathcal{R}$, $D^l(R) \subseteq C(R)$.*

Proof: This follows directly from Lemmas 1 and 2. $\square$

6 Applications

In this section, we consider, in more detail, each of the applications mentioned in the introduction.

6.1 Probabilistic (heterosexual) marriage problems

Let M be a set of men and W be a set of women. A *deterministic matching* either associates each person with a mate of the opposite sex or leaves them single.

As with other problems involving indivisibilities, randomization is one way to bring a sense of justice to a matching process (Aldershof et al. 1999, Klaus and Klijn 2006, Sethuraman et al. 2006). A common approach is to randomize only over the *ex post core* (or the *stable set*) (Sethuraman et al. 2006). However, if groups are able to commit to probabilistic allocations among themselves, a notion of *ex ante stability* is called for. That is, we should look for probabilistic matchings that are in the *core* with respect to their preferences over lotteries.[6]

To encode these problems in our model, let $N \equiv M \cup W$. For each $m \in M$, let $S_m \equiv \{m\} \cup W$ and for each $w \in W$, let $S_w \equiv \{w\} \cup M$. For each $i \in N$, let R_i be i's linear (von Neumann-Morgenstern) preferences over $\Delta(S_i)$.

The following is an implication of Theorem 2:

Corollary 1. *Every probabilistic marriage problem has a* lim-DIP allocation*.*

By considering preferences over lotteries, rather than just preferences over individual partners, we are able to account for intensities of preferences and achieve *ex ante* efficiency gains. The following example emphasizes this.

Example 3. *A probabilistic marriage problem where every* lim-DIP allocation Pareto dominates *the unique* stable *allocation.*

Let $M \equiv \{m_1, m_2\}$ and $W \equiv \{w_1, w_2\}$. Let their preferences over lotteries be such that they maximize the expectation of the index in the left hand column of the following table:

Index	R_{m_1}	R_{m_2}	R_{w_1}	R_{w_2}
1	w_1	w_2	m_2	m_1
0.1	m_1	m_2	w_1	w_2
0	w_2	w_1	m_1	m_2

The unique *ex-post* stable matching is $\pi^{xp} \in \Pi$ where for each $i \in N$, $\pi_i^{xp}(i) = 1$. For each $i \in N$, let $b_i \in S_i$ be i's most preferred (best) mate and l_i be i's least preferred (worst) mate. Then, for each $\pi \in D^l(R)$, $0.9 \geq \pi_i(b_i) \geq 0.1$ and $\pi_i(i) = 0$. Thus, π *Pareto dominates* π^{xp}. Note that this is not an unusual example. As long as for each $i \in N$, the intensity of preference for b_i over i is greater than the intensity of preference for i over l_i, each $\pi \in D^l(R)$ *Pareto dominates* π^{xp}. ◦

6.2 Trade under bilateral constraints

Suppose there is a group V of vendors of some good, and a group of buyers B. However, not every vendor can sell to every buyer. Instead, a graph $G \subseteq V \times B$ dictates which vendor-buyer pair can trade (Bochet et al. 2010): the pair $v \in V, b \in B$ can trade only if $(v, b) \in G$. Each $v \in V$ has single peaked preferences, $\hat{R}_v$, over the amount that he sells. Each $b \in B$ has single peaked preferences, $\hat{R}_b$, over the amount that he purchases. Since preferences are defined over the real line, we pick suitable bounds and normalize so that the maximum any buyer can

[6]See Manjunath (2011) for more on probabilistic marriage problems.

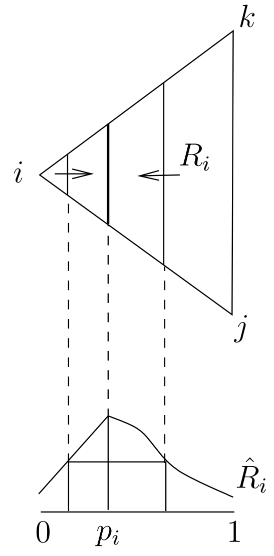

Figure 3: Let $i \in N$ be such that $S_i \equiv \{i, j, k\}$. We construct R_i from $\hat{R}_i$.

purchase or seller can sell is one unit.[7] The goal is then to specify an amount for each vendor to sell and for each buyer to purchase.

This model can be embedded in ours as follows: Let $N \equiv V \cup B$. For each $v \in V$, let $S_v \equiv \{v\} \cup B$ and for each $b \in B$, let $S_b \equiv \{b\} \cup V$. For each $i \in N$, let R_i be such that for each $\pi_i, \pi_i' \in \Delta(S_i)$, (see Figure 3)

$$\pi_i \mathcal{R}_i \pi_i' \Leftrightarrow \left(\sum_{j \in S_i} \pi_i(j) \right) \hat{R}_i \left(\sum_{j \in S_i} \pi_v'(j) \right).$$

The following is an implication of Theorem 2:

Corollary 2. *Every problem of trade under bilateral constraints has a* lim-DIP allocation.

Our model can accommodate two natural generalizations of these problems:

1. *Diverse vendors and buyers.* Corollary 2 holds for more general preferences on the part of both buyers and vendors. For instance, the vendors need not sell identical goods. The only restrictions on preferences are that, as listed earlier, they are continuous, convex, and locally non-satiated except at the maxima.

[7] While Bochet et al. (2010) do not assume that such a bound exists, if we apply their "voluntary participation" axiom, such a normalization is possible.

2. *More general graphs.* Rather than work with a bipartite graph such as G, Theorem 2 applies to a larger set of trading constraints. For instance, we can consider a situation where each person i owns an input that he can either sell to a set of buyers B_i or can combine with other inputs that he buys from the vendors V_i. Then, for each $i \in N$, and each $v \in V_i$, $i \in B_v$ and for each $b \in B_i, i \in V_b$. Thus, $S_i \equiv \{i\} \cup V_i \cup B_i$ and R_i is such that for each pair $\pi_i, \pi_i' \in \Delta(S_i)$,

$$\left.\begin{array}{c} \sum_{b \in B_i} \pi_i(b) \geq \sum_{b \in B_i} \pi_i'(b) \\ \text{and} \\ \sum_{b \in B_i} \pi_i(b) \geq \sum_{v \in V_i} \pi_i'(v) \end{array}\right\} \Rightarrow \pi_i \; R_i \; \pi_i'.$$

6.3 School Choice

Let S be a set of schools and C be a set of children. For each $c \in C$, let R_c be c's (more likely, his parents') *von Neumann-Morgenstern* preferences $\Delta(S)$. For each $s \in S$, let $\prec_s$ be a priority ordering of children for school s which involves large indifference classes. Let R_s be *von Neumann-Morgenstern* preferences over $\Delta(C)$ that are consistent with $\prec_s$. Call (R_C, R_S) an "augmented school choice problem."

We can now select an *lim-DIP* allocation.

Corollary 3. *Every augmented school choice problem has a* lim-DIP *allocation.*

In real-world school choice problem, ties are broken randomly (Erdil and Ergin 2008, Abdulkadiroğlu et al. 2010, Pathak and Sethuraman 2010) and used as inputs for deterministic algorithms like the *Boston* and *deferred acceptance* algorithms. Since these algorithms only consider ordinal information in students' preferences, there are *ex ante* efficiency losses (Abdulkadiroğlu et al. 2010). These losses can be avoided by modeling these problems as fractional matching problems.

Example 4. *A school choice problem.*[8]

Let $S \equiv \{s_1, s_2, s_3\}$ and $C \equiv \{c_1, c_2, c_3\}$. For each $s \in S$, let $\prec_s$ be degenerate so that each child has the same priority. Let R_C be defined by the following *von Neumann-Morgenstern* indices:

	u_{c_1}	u_{c_2}	u_{c_3}
s_1	0.8	0.8	0.6
s_2	0.2	0.2	0.4
s_3	0.0	0.0	0.0

Since each child has the same preferences over individual schools, both the *Boston* and *deferred acceptance* algorithms single out the same recommendation: equal

[8]Taken from Abdulkadiroğlu et al. (2010).

probability for each child at each school. This, however, is inefficient. Consider the allocation $\pi \in \Pi$ such that $\pi_{c_1 s_1} = \pi_{c_1 s_3} = 0.5$, $\pi_{c_2} = \pi_{c_1}$ and $\pi_{c_3 s_2} = 1$. Clearly π *Pareto-dominates* equal division (see Figure 4). Further, since for each $s \in S$, R_s is complete indifference, $\pi \in lim - DIP(R_C, R_S)$.

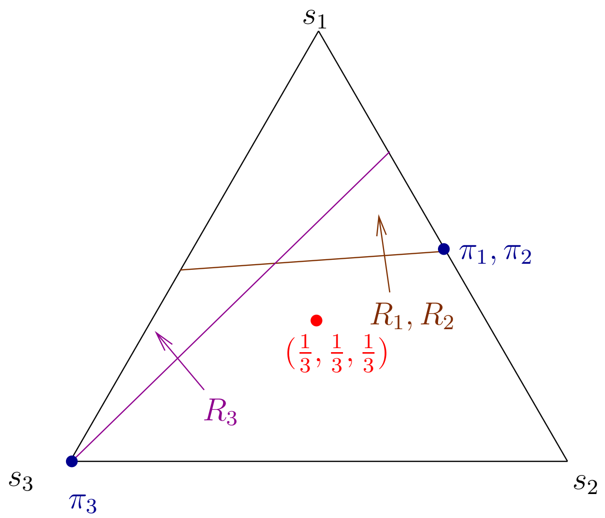

Figure 4: Clearly, equal division is dominated by π at (R_C, R_S).

6.4 Workers and employers

Let E be a set of employers and W be a set of workers. For each $e \in E$, let $\overline{R}_e$ be e's preferences over R_+^W. For each $w \in W$, let $\overline{R}_w$ be w's preferences over R_+^E. Each $w \in W$ has a unit supply of labor and each $e \in E$ can hire at most one unit of labor. The goal is to assign a work schedule to each worker. An allocation in the core of such a problem ensures participation of all groups. It is easy to see that, as in the applications above, this problem is a special case of our model.

The following is an implication of Theorem 2:

Corollary 4. *Every problem of workers and employers has a* lim-DIP allocation.

Example 5. *A problem involving workers and employers.*

Let $E \equiv \{e_1, e_2\}$ and $W \equiv \{w_1, w_2\}$. Let the preferences of each $e \in E$ be as shown in Figure 5. Let the preferences of each $w \in E$ be as shown in Figure 6. While there are many *lim-DIP allocations* for this economy, Figure 7 is an illustration of one of them.

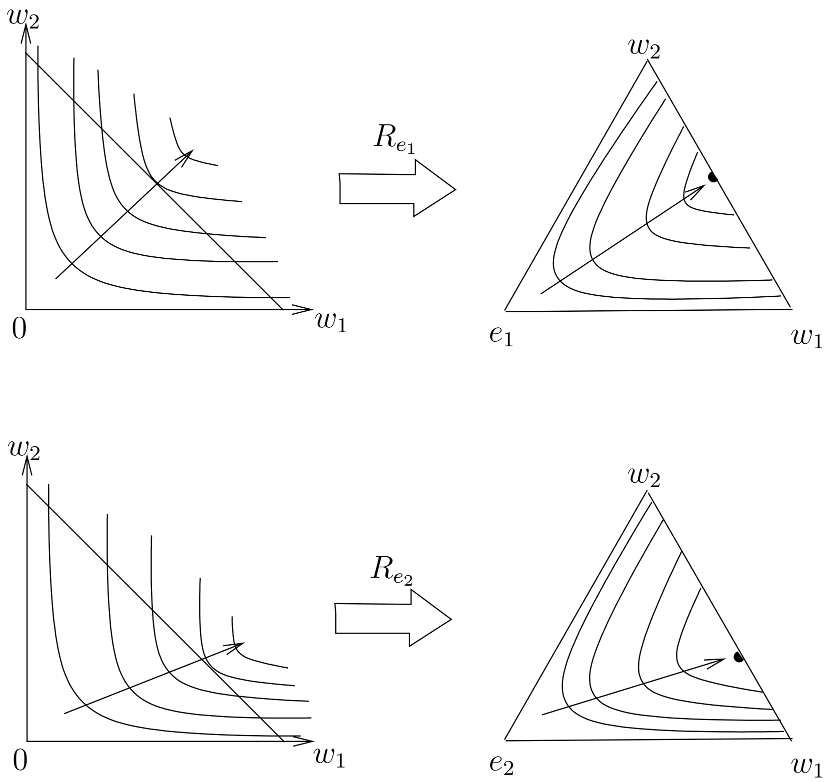

Figure 5: Preferences of e_1 and e_2.

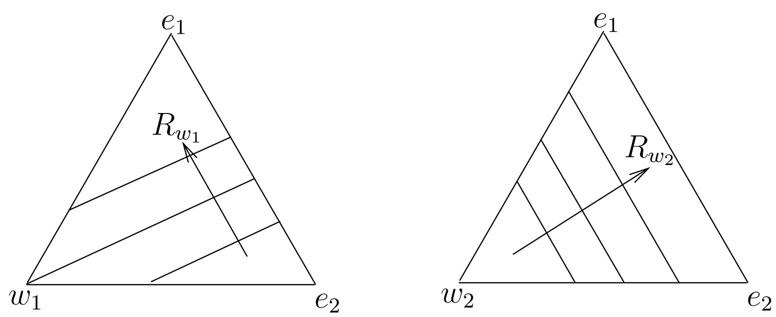

Figure 6: Preferences of w_1 and w_2.

7 Kinds of people

In this section, we describe a more general model than the one analyzed so far. In particular, we introduce the notion of a *kind* for each person and show that the

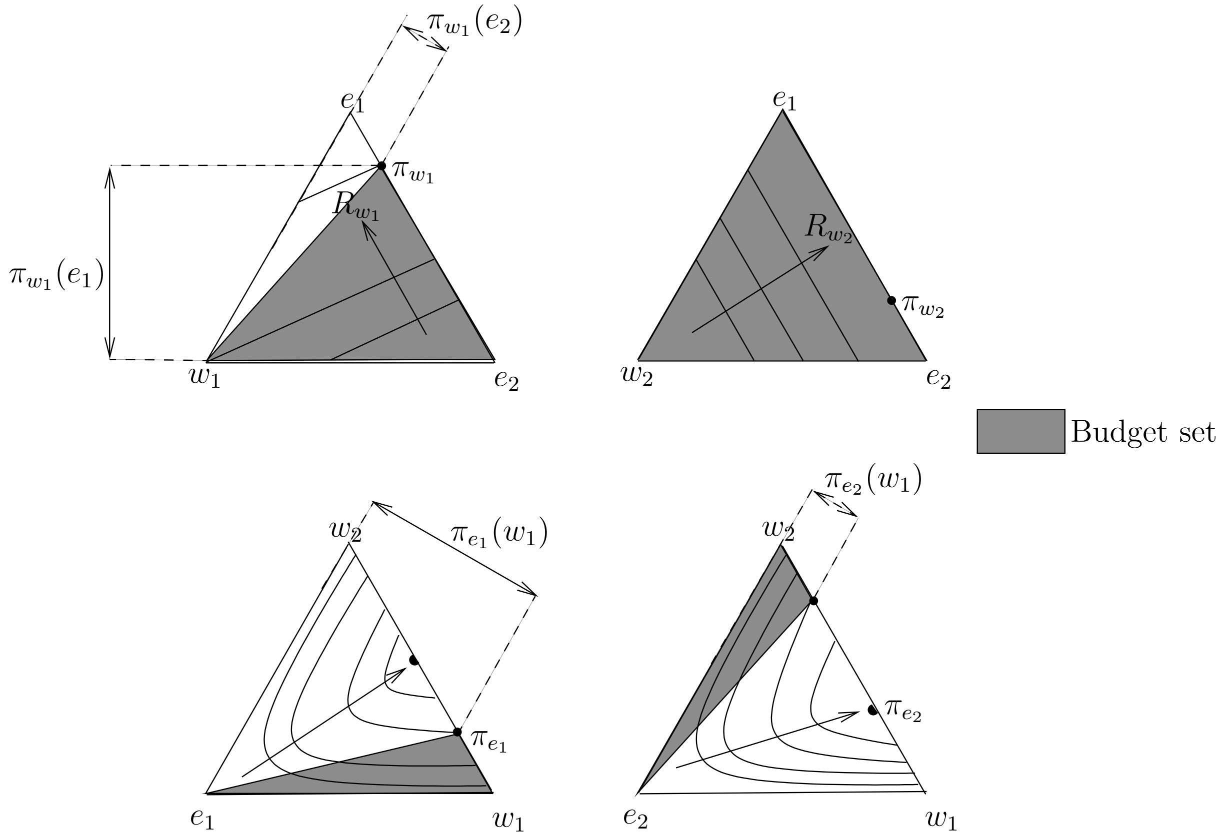

Figure 7: An lim-DIP equilibrium for the economy described in Figures 5 and 6 that can actually be supported by prices.

prices of an *lim-DIP* allocation can be indexed by *kinds* rather than identities.

Recall that the *lim-DIP* equilibria are somewhat like *Lindahl equilibria* as explained in Remark 1. A common indictment of *Lindahl* allocations, however, is that as the number of people involved increases, the number of prices must also increase. While prices are "personalized" in the definition of an *lim-DIP* equilibrium, we show here that they only need to be indexed by the *kind* of person and not his identity. The role of double-indexing is to reflect the "preferences of the resource." Suppose that two people are identical to the rest of the world. Since they are identical, anyone matched to them is indifferent between the two. The two should then face the same prices. Here, we generalize our earlier definitions and results to reflect this.

Let $\boldsymbol{K}$ be a set of *kinds*. For each $t \in K$, let $S_t \subseteq K$ be such that $t \in S_t$. The set of **potential partner kinds** of t are $S_t \setminus \{t\}$. As before, let N be the set of people involved. Let $\kappa \in K^N$ be such that for each $i \in N$, i's kind is κ_i. For each pair $s, t \in K$ if $s \in S_t$ then $t \in S_s$. For each $i \in N$, i's **consumption set** is $\Delta(S_{\kappa_i})$. Let $\boldsymbol{R_i}$, i's preference relation over $\Delta(S_{\kappa_i})$, be continuous, convex, and locally non-satiated except at its maxima. We require R_i to satisfy local non-satiation except at its maxima on $\Delta(S_{\kappa_i})$. Let $\mathcal{R}_i$ be the set of all such preferences. Let $(N_t)_{t \in K}$ be a partition of N such that for each $t \in K, N_t \equiv \{i \in N : \kappa_i = t\}$. An **economy** is described by a profile of preferences $R \in \mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$ and a profile of kinds $\kappa \in K^N$.

A **feasible allocation** specifies for each $i \in N$ a consumption bundle $\pi_i \in \Delta(S_{\kappa_i})$ in a way that for each $t \in K$ and each $s \in S_t$,

$$\sum_{\substack{i \in N_t \\ j \in N_s}} \pi_{ij} = \sum_{\substack{i \in N_t \\ j \in N_s}} \pi_{ji}.$$

Let $\mathbf{\Pi}$ be the set of feasible allocations.

Let $M \subseteq K \times K$ be such that $(s, t) \in M$ if and only if $s \in S_t$ and $t \in S_s$. An allocation $\pi \in \Pi$ is a **double-indexed price (DIP) allocation at** R if there is a vector $p \in \mathbb{R}^M_+$ such that for each $i \in N$,

$$\pi_i \in \underset{\pi'_i \in \Delta(S_{\kappa_i})}{\text{argmax}}\ R_i$$
$$\text{subject to}$$
$$\underbrace{\sum_{t \in S_{\kappa_i}} \pi'_{it} p_{\kappa_i t}}_{\text{Price of } \pi'_i} \leq \underbrace{\sum_{t \in S_{\kappa_i}} \sum_{j \in N_t} \pi'_{j\kappa_i} p_{t\kappa_i}}_{i\text{'s income at } \pi'},$$

and $\pi \in \Pi$. We interpret the price vector as follows: for each $(s, t) \in M$, p_{st} is the price that each $i \in N_s$ pays for a partnership with someone of kind t.

We refer to (π, p) as a **DIP equilibrium**. Let $\boldsymbol{D(R)}$ be the set of all *DIP allocations at R*.

Of course, for the same reasons as before, $D(R)$ may be empty. So we define *εDIP* and *lim-DIP* allocations here as well. While the definition of *εDIP equilibrium* is nearly the same as before, there are a few minor differences that we will highlight.

For each $i \in N$, $\Lambda(S_{\kappa_i}) \equiv \{\lambda \in \mathbb{R}^{S_{\kappa_i}}_+ : \sum_{t \in S_{\kappa_i}} \lambda_t \leq 1 + \varepsilon|N_{\kappa_i}|\}$. Clearly, for each $i \in N, \Delta(S_{\kappa_i}) \subset \Lambda(S_{\kappa_i})$. For each $R \in \mathcal{R}$ and each $\varepsilon \in (0, 1)$, let R^ε_i be an extension of R_i from $\Delta(S_{\kappa_i})$ to $\Lambda(S_{\kappa_i})$ such that R^ε_i is:

- *strictly monotonic* over $\{\lambda \in \Lambda(S_{\kappa_i}) : \sum_{t \in S_{\kappa_i}} \lambda_t < 1\}$,
- *continuous*,

- *convex*, and
- For each pair $x, y \in \Lambda(S_{\kappa_i})$ if $\sum_{t \in S_{\kappa_i}} x_t < \min\{1-\varepsilon, \sum_{t \in S_{\kappa_i}} y_t\}$, then $y \; P_i^\varepsilon \; x$. That is, if the sum of x's coordinates are less than $1 - \varepsilon$, then any point whose coordinates have a greater sum is preferred to x.

Claim 2. *Such R^ε exists.*

Proof: The proof is identical to that of Claim 1 with only a few changes. For each $x \in \Delta(S_{\kappa_i})$,

$$U(\hat{R}_i, x) \equiv \text{convex hull}\left(\left(1 - \frac{d(x)\varepsilon}{2}\right) M_i \cup U(\tilde{R}_i, x)\right) \cap \mathbb{R}_+^{S_{\kappa_i}}$$

Then, we complete the preference map over $\Lambda(S_{\kappa_i})$ in a way that indifference curves through any point x such that $\sum_{t \in S_{\kappa_i}} x_t \leq 1-\varepsilon$ are parallel to the simplex $\Delta(S_{\kappa_i})$. The remainder of the proof remains the same as that of Claim 1 and the extension of $\hat{R}$ to the positive orthant in the proof of Proposition 1. □

We say that $\pi \in \Pi$ is an **$\boldsymbol{\varepsilon}$DIP allocation** if there is $p \in \mathbb{R}_+^M$ such that for each pair $(s,t) \in M$,

$$p_{ss} + p_{tt} \geq p_{st} + p_{ts},$$

for each (s,t) such that $\sum_{\substack{i \in N_s \\ j \in N_t}} \pi_{ij} > 0,$

$$p_{ss} + p_{tt} = p_{st} + p_{ts},$$

for each $i \in N$,

$$\pi_i \in \underset{x_i \in \Lambda(S_{\kappa_i})}{\text{argmax}} \; R_i$$

subject to

$$\underbrace{\sum_{j \in S_i} x_{ij} p_{ij}}_{\text{Price of } x_i'} \leq \underbrace{(1-\varepsilon)p_{ii} + \left(\frac{\varepsilon}{|n|-1}\right)\left(\sum_{j \in N \setminus \{i\}} p_{jj}\right)}_{i\text{'s income}},$$

and for each pair $(i,j) \in M$,

$$\sum_{\substack{i \in N_t \\ j \in N_s}} \pi_{ij} = \sum_{\substack{i \in N_t \\ j \in N_s}} \pi_{ji}.$$

The "clearing" condition here is more permissive. The allocation π need not a feasible allocation itself. However, it is "within ε" of one.

Proposition 2. *For each* $\varepsilon \in (0,1)$ *and each* $(R,\kappa) \in \mathcal{R} \times K^N, \varepsilon D(R,\kappa) \neq \varnothing$.

Proof: This proof is very similar to that of Proposition 1. Once preferences are extended to the positive orthant, and the problem is encoded as an Arrow-Debreu model. For each $i \in N$, i's consumption space is

$$X_i \equiv \mathbb{R}_+^{S_{\kappa_i}} \times \{0\} \subset R_+^M.$$

Define the extension $\overline{R}_i^\varepsilon$ of R_i^ε to X_i exactly as in the proof of Proposition 1. For each $x \in \Lambda(S_{\kappa_i})$, set

$$U(\overline{R}_i^\varepsilon, x) \equiv \text{comp } U(R_i^\varepsilon, x).$$

Define M_i as before and and translate the preference map over $U(\overline{R}_i^\varepsilon, M_i)$.

Firms are defined the same way, except that the are indexed by pairs of kinds rather than pairs of people. The production set of firm $(s,t) \in F$ is

$$Y_{\{s,t\}} \equiv \left\{ y \in \mathbb{R}^{\{st,ts,ss,tt\}} : y_{st} = y_{ts} = -y_{ss} = -y_{tt} \right\} \times \{0\} \subset \mathbb{R}^M.$$

For each $i \in N$, i's endowment is $\omega^i \in \mathbb{R}_+^M$ such that for each pair $s,t \in K$,

$$\omega_{st}^i = \begin{cases} 1-\varepsilon & \text{if } s=t=\kappa_i, \\ \frac{\varepsilon}{|N|-1} & \text{if } j=k\neq\kappa_i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For each $\{s,t\} \in F$, and each $i \in N$, let $\sigma_i(\{s,t\})$ be i's share of $\{s,t\}$. Thus, for each $\{s,t\} \in F, \sum_{i\in N} \sigma_i(\{s,t\}) = 1$.

As before, the economy $E \equiv (X, Y, \overline{R}^\varepsilon, \omega, \sigma)$ has a *competitive allocation* $(x,y,p) \in X \times Y \times R_+^M$. We show that (x,p) is actually an *εDIP equilibrium.* Since (x,y,p) is a *competitive equilibrium*, for each $i \in N$, $x_i \; \overline{R}_i^\varepsilon \; (1-\varepsilon)\delta_i$. Then, by definition of R^ε and therefore $\overline{R}^\varepsilon$, for each $i \in N$, $\sum_{t\in S_{\kappa_i}} x_{it} \geq 1-\varepsilon$. Thus, by feasibility, for each $t \in K$ and $i \in N$, $\sum_{t\in S_{\kappa_i}} x_{it} \leq 1+\varepsilon|N_t|$. Finally, as argued in the proof of Proposition 1, by definition of Y and feasibility,

$$\sum_{\substack{i \in N_t \\ j \in N_s}} \pi_{ij} = \sum_{\substack{i \in N_t \\ j \in N_s}} \pi_{ji}.$$

□

As before, an allocation $\pi \in \Pi$ is a **limit DIP (lim-DIP) allocations** if there is a sequence $\{\pi^\varepsilon\}_{\varepsilon\in(0,1)} \in \Pi$ such that

$$\begin{array}{rl} i) & \text{for each } \varepsilon \in (0,1), \pi^\varepsilon \in D^\varepsilon(R) \text{ and} \\ ii) & \lim\limits_{\varepsilon\to 0} \pi^\varepsilon = \pi. \end{array}$$

Let $\boldsymbol{D^l(R)}$ be the set of all *limit DIP allocations.*.

From Proposition 2 we have the following.

Theorem 4. *For each* $(R,\kappa) \in \mathcal{R}^N \times K^N$, $D^l(R,\kappa) \neq \varnothing$.

7.1 Discussion regarding kinds

There are two distinct benefits to adding kinds to our model. The first is that two people who are, for all intents and purposes, the same should be given the same opportunities. The *lim-DIP* solution does exactly that. Since prices are indexed by kind rather than identity, each person of a particular kind is faced with exactly the same "budget set." If prices are indexed by identities, then identical people may be treated differently.

The second is to apply our model to situations where no person is unique in the eyes of others. Take, for instance, a school district where each school has a large number of seats and large groups of students have identical priorities at each of the schools. Or think of a problem involving many workers and many tasks where there are many workers with identical skills and many tasks that are identical.

Unlike *Lindahl equilibria*, as the number of people involved increases, as long as there number of kinds remains fixed, the dimension of the price vector remains fixed. Since ε*DIP* equilibria are actually *general equilibria* of appropriately defined Arrow-Debreu models for fixed utility representations, the gains from misreporting preferences diminish as the number of people of each kind increases.

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