Endogenous fertility policy and unfunded pensions

Alison Booth
The Australian National University
and
University of Essex

Facundo Sepulveda
Universidad de Santiago de Chile [1]

February 3, 2012

[1]Corresponding author: Facundo Sepulveda, Department of Economics, FAE. Universidad de Santiago de Chile. Alameda Libertador B. O'Higgins 3363. Santiago. Chile. Email: facundo.sepulveda@fsp.cl. We wish to thank Gerhard Glomm, Steve Dowrick, Andreas Engel and Abhinay Muthoo for useful suggestions to improve the manuscript. All remaining errors are ours.

Abstract

We study the joint determination of fertility subsidies and Social Security taxes in an overlapping generations model where agents are heterogeneous in endowments. In equilibria where Social Security is valued, old and poor young agents form a coalition that sustains Social Security. When voting for fertility subsidies, the young take into account both the deadweight loss of such subsidies and the gains from a higher future tax base. They also take into account a third effect of increasing population growth: that of a decrease in future Social Security benefits as a consequence of a change in the identity of the future decisive voter.
Keywords: Political economy, OLG models, social security, endogenous fertility, redistribution.
JEL Classification: E62, H2, H30, H55, J13, J14.

1 Introduction

In this paper we develop a political economy model of the joint evolution of unfunded Social Security (SS) systems and fertility policies. We use it to investigate the viability of unfunded Social Security and the role of fertility in securing it. We employ an overlapping-generations framework with endogenous fertility in which agents vote on the size of fertility subsidies, on whether or not to continue with the SS system, and on the magnitude of pension taxes and payments. It is well known that a number of different policies - and in particular policies related to public education and immigration - can make SS viable in situations where otherwise it would be abandoned. However, our framework adds significantly to the literature by exploring the way in which current fertility subsidies can affect the identity of the future decisive (median) voter, and thereby affect future pension/SS entitlements. This has not been investigated before. And yet as we discuss below there are historical episodes in which fertility policies have been designed to alter the identity of future decisive voters.

A form of strategic manipulation of tomorrow's constituency by means of current public policies is present in Hassler et al. [2003]. In that paper, redistributive policies reduce the incentives to invest in human capital, creating future constituencies that are more inclined to vote for the continuation of the welfare state. Like that paper, we are interested in the conflict within and between generations over redistribution, and the conditions under which the welfare state will survive. However, unlike Hassler et al. [2003], we focus on the role that fertility policies may have in ensuring the survival of the Social Security system, which introduces an extra public policy decision to be made. Also, while in Hassler et al. [2003] young and old agents are homogeneous, we allow for a continuum of ability levels, and study the role of income heterogeneity in the sustainability of the Social Security system.

A number of contributions on the political economy of SS (see Galasso and Profeta [2002] for a survey) are also relevant for our purposes. First, Casamatta et al. [2000] and Tabellini [2000] study SS as a device that not only redistributes income from young to old, but also from wealthy to poor households, and therefore SS arrangements can be sustained as a political equilibrium without resorting to intergenerational (e.g. dynamic efficiency) considerations. Cooley and Soares [1999] and Galasso [1999] study SS as an institution with inherited rules that are costly to change, but their approaches are quite different from the one taken here. The paper that is closer to ours is Boldrin and Rustichini [2000]. In that model, agents are confronted with an existing promise of paying a SS tax to old agents, and may either abandon the SS system altogether, or pay the SS tax and vote for a level of SS benefits in the next period. The analysis in Boldrin and Rustichini [2000] gives an explicit dynamic dimension to the problem of sustaining a SS system, and provides a clear interpretation of it as one of unfunded SS liabilities, in line with most of the public policy debate.

A small number of studies examine the joint determination of SS along with another dimension of the welfare system, as we do. In particular, Conde-Ruiz and Galasso [2003] separate the redistributive aspect of SS into within and between cohort components, and consider the circumstances under which both aspects arise as an equilibrium. Boldrin and Montes [2002] show that unfunded SS creates incentives for the optimal provision of public education, a result generalized by Rangel [2003]. Finally, Kemnitz [2000] and Poutvaara [2003] study the joint determination of unfunded SS and education subsidies. We are aware of no paper that studies the joint determination of fertility policies and SS systems.

Our paper extends the literature on the political economy of SS systems to incorporate the endogenous determination of fertility policies. We find that strategic setting of fertility subsidies to limit the political influence of the newborn generation in the future decreases the levels of both fertility subsidies and SS taxes. We also obtain the result that the existence of fertility subsidies is a necessary condition for the existence of a unique, positive and globally stable steady state level of SS taxes.

This paper has three other sections. In the next section we present the model, and define the equilibrium in which we are interested. In section 3 we derive and present the results with endogenous policy variables, and in section 4 we conclude.

2 The voting equilibrium

We study an overlapping generations economy where households live for two periods. Households have preferences defined over their consumption when young c^y, the number of children to be born at the end of the first period n, and their consumption when old c^o.

An endowment of α_i is received when young and, as there is no technology to allow saving, agents must rely on SS transfers to consume when old. Households are heterogeneous in their endowments (α), with the distribution of α over young households summarized by the uniform CDF $G(\alpha)$, with mean θ and support $\Theta \equiv [\underline{\alpha}, \overline{\alpha}]$.[1] In this model income heterogeneity serves two purposes. First, it allows us to investigate both within generation and between generation redistribution, which turns out to have important implications for our results. Second, it ensures that Social Security is valued even if the economy is dynamically efficient, which we believe introduces a stronger motivation for the existence of the Social Security system.

There are three policy instruments, SS taxes τ^{ss}, fertility subsidies τ^f, and the option to keep vs abandon the SS system λ. The SS system is unfunded, or pay-as-you-go, so young households pay a proportional tax on their endowment, which finances the unique level of SS benefits received by old households at every period. Fertility subsidies (taxes) are designed to reduce (increase) the cost of having children bn, where b is the cost per child, and are financed by a lump sum tax (rebate) T.[2] Young household i at time t maximizes the direct utility function in (2) in which taxes and subsidies are exogenous , subject to the per-period budget constraints.

$$\begin{aligned} \max_{\{c^y, c^o, n\}} \quad & \hat{V}^y(c_t^y, c_{t+1}^o, n_t) = c_t^y + \beta c_{t+1}^o \quad + \gamma \ln n_t \qquad (2) \\ s.t. \quad & (1 - \lambda_t \tau_t^{ss})\alpha_i = c_t^y + b(1 - \tau_t^f) n_t + T_t \\ & c_{t+1}^o = \lambda_{t+1} ss_{t+1}. \end{aligned}$$

Here, ss_{t+1} represents SS benefits at time $t+1$ and β is a time discount factor. Note that SS taxes at t are paid only if the SS system is kept in that period ($\lambda_t = 1$), and benefits at $t+1$ are received if the system is kept at $t+1$ ($\lambda_{t+1} = 1$). Old households make no meaningful private decisions, and simply consume their SS benefits, if any: $\hat{V}^o = \lambda_t ss_t$

The use of quasilinear preferences, together with a lower bound on individual endowments α_i will ensure that the choice of n is independent of wealth, which simplifies aggregation greatly.

[1] This implies:

$$G(\alpha_i) = \begin{cases} 0 & \text{if } \alpha_i < \underline{\alpha} \\ \frac{\alpha_i - \underline{\alpha}}{\overline{\alpha} - \underline{\alpha}} & \text{if } \alpha_i \in [\underline{\alpha}, \overline{\alpha}] \\ 1 & \text{if } \alpha_i > \overline{\alpha} \end{cases} \qquad (1)$$

[2] Note that n is normalized so that the unit of measurement of the population is the young (two person) household, so $n = 1$ implies 2 children per couple.

It is important to note that the private decisions made by each household are taken after the public decisions. Moreover, since there is a continuum of agents, their individual decisions do not affect the government's budget balance. This is why, in problem (2), taxes and subsidies are exogenously given to households.

In this arrangement, all households choose the same number of children n_t. Fertility and consumption when young and old follow [3]

$$\begin{aligned} n_t &= \frac{\gamma}{b(1-\tau_t^f)}. && (3) \\ c_t^y &= (1-\lambda_t \tau_t^{ss})\alpha_i - T_t - \gamma && (4) \\ c_{t+1}^o &= \lambda_{t+1} ss_{t+1} && (5) \end{aligned}$$

The government raises taxes and provides subsidies and SS benefits under a restrictive rule of budget balance: SS benefits are financed with SS contributions, and fertility subsidies are financed through a lump sum tax paid only by the young. These conditions are formalized below.

$$\begin{aligned} N_t \tau_t^{ss} \theta &= N_{t-1} ss_t && (6) \\ T_t &= b n_t \tau_t^f && (7) \end{aligned}$$

Where N_t is the number of young at t. Because endowments are exogenous and T is lump sum, both the fertility and SS programs are financed in a non distortionary fashion. However, as ss_{t+1} is the same for every old household, but SS taxes are paid according to wealth, the SS system is redistributive. Fertility subsidies on the contrary are distortionary in that they affect the number of children, but do not imply a redistribution of income. The redistributive aspect of SS and the distortionary aspect of fertility subsidies can therefore be isolated in this setting.[4]

The model is closed by specifying the law of motion for population. The population of young household evolves according to

$$N_{t+1} = n_t N_t. \tag{8}$$

[3]We restrict the parameters to ensure an interior solution where consumption when young is strictly positive. The condition on the parameters is that disposable income when young is larger than γ:$\gamma < (1-\tau_t^{ss})\underline{\alpha} - T_t$

[4]The non distortionary nature of the financing of both programs is a natural consequence of having an endowment economy, where the only meaningful private decision is that of consuming versus having children. At the same time, the separation of the SS and fertility budgets in this framework is a way to ensure the self-financing of the fertilitity program within each household.

Equations (3) to (8) characterize the competitive equilibrium. Once the individual decision rules (3), (4) and (5) are substituted into the utility functions for young and old agents, we obtain the indirect utility functions. Making use of the budget constraints (6) and (7), we finally obtain the reduced indirect utility functions:

$$
\begin{aligned}
V_i^y(\tau_t^f, \tau_{t+1}^{ss}, \lambda_t) &= (1 - \lambda_t \tau_t^{ss})\alpha_i + \frac{\lambda_{t+1}\theta\beta\gamma\tau_{t+1}^{ss}}{(1-\tau_t^f)b} - \frac{\gamma}{1-\tau_t^f} \\
&\quad + \gamma \ln \frac{\gamma}{(1-\tau_t^f)b}. \qquad (9) \\
V^o(\lambda_t) &= \lambda_t \frac{\theta w \tau_t^{ss}\gamma}{(1-\tau_{t-1}^f)b}. \qquad (10)
\end{aligned}
$$

Equations (9) and (10) represent the preferences of young and old households over the policy choices (τ_{t+1}^{ss},τ_t^f,λ_t). Note that we use hats to differentiate direct and indirect utility functions utility in (2) and (9).

2.1 The voting game

In our political setup, decisions are taken by majority voting. At each time period, agents vote over three dimensions: the continuation versus abandoning of Social Security (λ_t), the Social Security tax for the next period (τ_{t+1}^{ss}), and the current period fertility subsidy (τ_t^f). We assume sincere voting: individuals vote for their preferred choices even if this will not change the equilibrium outcome. Three issues must be considered before we define a voting equilibrium.

First, voting occurs over three dimensions, and it is well known that in such cases voting cycles may arise. Our approach is to focus on a structure induced equilibrium (see Shepsle [1979]). This equilibrium concept reduces to finding the fixed point of three functions: $\tau_{t+1}^{ss*}(\tau_t^f, \lambda_t)$, $\tau_t^{f*}(\tau_{t+1}^{ss}, \lambda_t)$, and $\lambda_t^*(\tau_{t+1}^{ss}, \tau_t^f)$, where each function maps values of two policy parameters into the median voting outcome of the third. This equilibrium has then the desirable property that, given any two equilibrium policies, the third would be the one chosen by majority voting.

Second, we focus on subgame perfect equilibria. Subgame perfection disciplines the choice of young voters with respect of future SS taxes, as they know that such tax levels must not be so high as to induce future voters to abandon SS altogether. When making public choices, voters need to anticipate the values of future policy variables. Given a candidate equilibrium, we assume that voters solve the one period game where there is commitment over the future values of these variables. In equilibrium, such expectations must be fulfilled.

Finally, we must address the question of what prevents young voters from defaulting on their SS obligations, and then reinstall SS when old. We follow Conde-Ruiz and Galasso [2003] and Boldrin and Rustichini [2000] in introducing an implicit contract between adjacent generations and examining whether it would support the institution of SS. This implicit contract could arise from the belief by the young that their commitment to supporting SS will be rewarded with a similar attitude by next period voters. Formally, we focus on trigger strategies where the young at t vote for $\lambda_t = 0$ if, at any time $s < t$, $\lambda_s = 0$ was the voting outcome over this parameter. Note that such trigger strategies are subgame perfect: if all young households except for household i vote $\lambda_t = 0$, household i will not be decisive, so his payoff will be the same regardless of his vote on λ_t. Appendix A.1 presents a formal description of the game, including the equilibrium.

3 Properties of the equilibrium

In this section we examine the properties of the equilibrium. Our aim is to illustrate the two main tradeoffs present in the voting decisions. The first tradeoff involves the gains from increased fertility in the form of a larger tax base in the future, weighted against the deadweight loss from subsidizing fertility. The second tradeoff involves weighting the net gains from increased fertility via a larger tax base, as discussed above, against a lower level of SS taxes that tomorrow's decisive voter will be willing to pay.

3.1 Preferences over public choices

In what follows, we characterize voting by young and old households. An important first result is that there is no conflict of preferences over the levels of tax rates:

Lemma 1 *All (young) voters choose the same tax rates.*

Proof: note that because V_i^y is separable in α_i, the problem of choosing tax rates once SS is continued can be stated without α_i being an argument of the objective function or the constraints (see appendix A.2).

The reason why all young agents vote for the same fertility subsidy is that they all have the same fertility behavior, and the costs of subsidizing fertility are equally similar across young agents. At the same time, because the Social Security tax chosen is that which will be paid by the young in the next period. Young households will face a trade off in the choice of fertility subsidies, as these increase next period SS payments, but are otherwise distortionary. With regards to the SS tax for next period, if $\lambda_t = 1$ young voters' welfare is monotonically increasing in τ_{t+1}^{ss}. This Social Security tax level is independent of the identity of the young agent.

Old households (expression (10)) are indifferent among different levels of the tax rates chosen in the current period $\{\tau_{t+1}^{ss}, \tau_t^f\}$, as these choices will only have consequences in the next period. Since they are indifferent, the old abstain from voting on tax rates.

Regarding the choice over the continuation vs. abandoning of Social Security, note that the old will always prefer to keep SS, as they stand to lose their benefits otherwise. The young in turn will be divided: as SS contributions are proportional to income, but benefits are not, the poorest among the young gain, while the wealthier lose, from the institution of Social Security. This is formalized in the following lemma.

Lemma 2 *Voting over the existence of Social Security.*

1. *The old vote $\lambda_t = 1$*

2. *There exists an endowment level α^m such that the young vote $\lambda_t^i = 1$ if $\alpha^i \leq \alpha^m$ and $\lambda_t^i = 0$ otherwise.*

Proof: for the fist point, note that the old will only receive SS benefits if SS is continued, so they vote $\lambda^m = 1$.

To prove the second point, note that the value of continuing SS for young household i is

$$V_i^y(\tau_t^f, \tau_{t+1}^{ss}, \tau_t^{ss}) - V_i^y(0,0,0) = -\tau_t^{ss}\alpha_i + \frac{\theta\beta\gamma\tau_{t+1}^{ss}}{(1-\tau_t^f)b} - \frac{\gamma\tau_t^f}{1-\tau_t^f} - \gamma\ln(1-\tau_t^f) \tag{11}$$

Which is decreasing in the endowment α_i and is negative for all endowments larger than some critical value α^m.

The above lemma implies that the SS system is always sustained by a coalition of the old (who are always beneficiaries) and the poor young. The old favor SS because, for them, all the costs associated with it are sunk. The poor young, in turn, favor SS because they gain from the redistributive aspect of the system.

A further important property of the equilibrium is that the decisive voter obtains no surplus from the institution of SS.

Lemma 3 *The decisive voter, with endowment α^m, is indifferent between the allocation with and without SS: Half of the voters choose $\lambda = 1$ and the other half $\lambda = 0$.*

Proof: the proof proceeds by contradiction. If the decisive voter at $t+1$ receives a net surplus from continuing SS, the decisive voter at t could have increased next period's SS taxes (τ^{ss}_{t+1}) by a small amount, and SS would still be continued at $t+1$. If on the contrary, the decisive voter at $t+1$ receives a net loss from continuing SS, he will vote $\lambda^m = 0$, so the decision to continue SS by the decisive voter at t could not have been derived from an equilibrium policy.

The fact that the decisive voter is indifferent between continuing or abandoning SS allows us to obtain the equilibrium tax rates $\{\tau^{ss}_{t+1}, \tau^f_t\}$ by solving a program where households maximize their indirect utility subject to the constraint that the decisive voter next period will be indifferent in his choice over λ.

Using lemma 2 we can obtain an expression for the endowment level of the decisive voter. Note that old voters will unanimously choose to honor the SS promise ($\lambda_{t+1} = 1$), since they are the beneficiaries. If we normalize the number of old households to 1, then the mass of voting households is $1 + n_t$. With $n_t > 1$, the decisive voter household is such that a proportion $\frac{n_t - 1}{2n_t}$ of young households will vote $\lambda_{t+1} = 1$ [5]. Together with the fact that the poorest young households are the ones to vote for continuing SS, this implies that the decisive voter household at time $t+1$ will have an endowment level

$$\alpha_{m,t+1} = \underline{\alpha} + (\overline{\alpha} - \underline{\alpha})\frac{n_t - 1}{2n_t}. \tag{12}$$

[5] If $n_t > 1$, the decisive voter household for the two generations is such that $(1 + n_t)/2$ vote for the same choice. Thus the proportion of the young voting the same way as the old is given by $[\frac{1+n_t}{2} - 1]/n_t = \frac{n_t - 1}{2n_t}$.

Regarding the interactions between the SS system and fertility subsidies, note that the only motivation to vote for fertility subsidies comes from the existence of unfunded SS. At the same time, it is the heterogeneity in endowments that makes Social Security valued in equilibria that are dynamically efficient. The following lemma formalizes these results.

Lemma 4 *Interactions between SS and fertility subsidies.*

1. *Social Security is necessary for fertility subsidies to be valued*

2. *If the economy is dynamically efficient, heterogeneity in endowments is necessary for Social Security to be valued.*

Proof: For point 1, the net value of fertility subsidies in the absence of Social Security (from expression (11)) is

$$-\frac{\gamma \tau_t^f}{1-\tau_t^f} - \gamma \ln(1-\tau_t^f) \tag{13}$$

with a maximum at $\tau_t^f = 0$.

For point 2, note that if endowments are homogeneous and the economy is dynamically efficient, the young cannot be made better off by trading across generations.

We present the equilibrium in three steps. First, we derive the evolution of SS in an economy without fertility subsidies. Then, we introduce fertility subsidies but examine the outcomes when agents do not internalize the effects of their choices on the identity of the future decisive voter. Finally, we study the model with fully rational agents.

3.2 No fertility subsidies

It is instructive to consider initially the model with no recourse to fertility subsidies. Our algorithm for finding the equilibrium proceeds by first deriving the tax rates chosen if SS is not abandoned, so that $\lambda_t = 1$ for all t, and then considering the choice of keeping the SS system. The algorithm reduces to having young households choose $\{\tau_{t+1}^{ss}\}$ by maximizing their (reduced) indirect utility function subject to the incentive compatibility constraint that next period's decisive voter will be (weakly) better off keeping the Social Security system.

$$\max_{\tau^{ss}_{t+1}} \quad V^y(0, \tau^{ss}_{t+1}, \tau^{ss}_t) = (1 - \tau^{ss}_t)\alpha_i + \tfrac{\theta\beta\gamma\tau^{ss}_{t+1}}{b} - \gamma \quad +\gamma \ln \frac{\gamma}{b} \tag{14}$$

$$s.t. \qquad -\tau^{ss}_{t+1}\alpha_m + \tfrac{\theta\beta\gamma\tau^{ss}_{t+2}}{b} \geq 0 \qquad \text{if } \gamma/b > 1 \tag{15}$$

$$\tau^{ss}_{t+1} \leq 1 \qquad \text{otherwise.} \tag{16}$$

With the decisive voter having an endowment $\alpha_m = \underline{\alpha} + (\overline{\alpha} - \underline{\alpha})\frac{\gamma/b-1}{2\gamma/b}$. Constraint (15) applies in case of an interior solution, when next period's decisive voter is also a young agent. The constraint states that, to next period's decisive voter, the value of keeping SS (or $V^y_i(0, \tau^{ss}_{t+2}, \tau^{ss}_{t+1})$ as defined in expression (9)) must be at last as high as the value of abandoning SS (or $V^y_i(0,0,0)$). Note that in this program τ^f_{t+1} is set to zero, as we are considering the case without fertility subsidies. Constraint (16) in turn represents a corner solution: if $n_t = \gamma/b \leq 1$, the current young will be in a majority in the next period, and therefore can set a expropriatory tax rate of $\tau^{ss}_{t+1} = 1$. In what follows we disregard this possibility and consider only interior solutions, where constraint (15) applies. The optimal choice for the SS tax rate is obtained from standard first order conditions:

$$\tau^{ss}_{t+1} = \frac{\theta}{\alpha_m}\frac{\gamma}{b}\beta\tau^{ss}_{t+2} \tag{17}$$

We can decompose the coefficient multiplying τ^{ss}_{t+2} into three parts: $\frac{\theta}{\alpha_m}$, $\frac{\gamma}{b}$, and β. Note that the ratio of average income to the income of the richest household voting for SS, denoted by $\theta/\alpha_m > 1$, is a measure of how redistributive the SS system is. Note also that the ratio γ/b is both the fertility rate and the dependency ratio in the next period. The intuition behind (17) is as follows: The larger θ/α_m is, the more do low endowment young households at $t+1$ have to gain from SS for any given tax rate τ^{ss}_{t+2}. Consequently, they in turn can be taxed more and still want to preserve SS. A similar argument holds for the fertility rate, as a higher rate implies larger SS benefits for any given tax rate, since there are more young households to be taxed. Finally, since costs τ^{ss}_{t+1} are paid one period before SS benefits for the decisive voter at $t+1$, these benefits need to be discounted by β.

The law of motion for SS taxes can be obtained by inverting and lagging equation (17).

$$\tau^{ss}_{t+1} = \tau^{ss}_t \frac{\alpha_m b}{\theta\beta\gamma} \tag{18}$$

Since the law of motion for SS taxes is linear, in general an interior steady state will not exist. This result is similar to that in Boldrin and Rustichini [2000], who find that zero is the only stable steady state, and that therefore any SS system will gradually shrink as time progresses.

In this version of our model, a high fertility rate, high inequality, and high degree of patience (β close to one), all contribute to making SS implementable ($x < 1$) since they imply that any level of future SS taxes is associated with higher levels of benefits. Successive decisive voters may therefore prefer to continue the SS system even if the sequence of SS tax rates is decreasing.

3.3 Fertility subsidies

We now allow for voting over fertility subsidies. We begin by noting that fertility subsidies affect the identity of the decisive voter in the next period, as described in the following lemma.

Lemma 5 *Higher (lower) fertility subsidies at t imply a higher (lower) endowment decisive voter at $t+1$.*

Proof: Expression (12) shows the endowment level for the decisive voter household at time $t+1$:

$$\alpha_{m,t+1} = \underline{\alpha} + (\overline{\alpha} - \underline{\alpha})\frac{n_t - 1}{2n_t}. \tag{19}$$

By (3), we have $\frac{\partial n}{\partial \tau^f} > 0$, so $\frac{\alpha_{m,t+1}}{\tau_t^f} > 0$

This result is illustrated in figure 1, where the endowment level is measured on the vertical axis. The horizontal axis represents the number of voters, where the number of old households is normalized to one and young households are ranked from poorest to wealthiest. As fertility increases from n to n', the endowment level of the decisive voter household increases from α_m to α'_m.

We examine an equilibrium with valued SS. Using the expression for α_m in (12), and with n_t given by (3), the equilibrium tax rates can be obtained as the solution to the following program:

$$\max_{\{\tau_t^f, \tau_{t+1}^{ss}\}} \quad V^y(\tau_t^f, \tau_{t+1}^{ss}, 1)$$
$$= (1 - \tau_t^{ss})\alpha_i + \tfrac{\theta\beta\gamma\tau_{t+1}^{ss}}{(1-\tau_t^f)b} - \tfrac{\gamma}{1-\tau_t^f} \qquad + \gamma \ln \frac{\gamma}{(1-\tau_t^f)b} \tag{20}$$

$$s.t.\quad -\tau^{ss}_{t+1}\{\overbrace{\frac{(1-(1-\tau^f_t)b/\gamma)(\overline{\alpha}-\underline{\alpha})}{2}+\underline{\alpha}}^{\alpha_{m,t+1}}\} \quad +\frac{\theta\beta\gamma\tau^{ss}_{t+2}}{(1-\tau^f_{t+1})b}$$

$$-\tfrac{\gamma}{1-\tau^f_{t+1}}+\gamma+\gamma\ln\tfrac{1}{1-\tau^f_{t+1}}=0 \qquad \text{if } \tau^f_t>1-\gamma/b \tag{21}$$

This problem is similar to (14)-(15) except that households can now vote over the level of τ^f_t, and therefore can affect the identity of the decisive voter in the next period. Expression (21) then represents $V^y_i(\tau^f_t,\tau^{ss}_{t+1},1)\geq V^y_i(0,0,0)$ for a decisive voter with endowment $\alpha_{m,t+1}$, as defined in expression (12).

We discuss the results in two steps. First, we derive the equilibrium in a model where agents do not anticipate the effects of their actions on the identity of the future decisive voter. This myopic version of the equilibrium will allow for a discussion of the first tradeoff in the choice of fertility subsidies: that of a higher tax base and therefore larger future SS benefits, against the deadweight loss of the subsidy. We then discuss the equilibrium in the model with fully rational voters, which will allow for a discussion of the second effect: that of lower future SS benefits due to a change in the identity of the decisive voter.

3.3.1 Myopic voters

In this subsection we study a model with myopic voters. In this equilibrium, derived formally in appendix A.3, voters solve the problem in (20) to (21) taking $\alpha_{m,t+1}$ in equation (21) as parametric. That is, they do not internalize the effect of their chosen fertility policy on the identity of next period's decisive voter. In an interior solution, with μ denoting the multiplier assigned to constraint (21), tax rates are obtained from the following first order conditions:

$$(\tau^{ss}_{t+1})\quad \tfrac{\theta\beta\gamma}{(1-\tau^f_t)b}-\mu\{\underline{\alpha}+\tfrac{(\overline{\alpha}-\underline{\alpha})}{2}(1-(1-\tau^f_t)b/\gamma)\}=0 \tag{22}$$

$$(\tau^f_t)\quad -\tfrac{\gamma}{(1-\tau^f_t)^2}+\tfrac{\theta\beta\gamma\tau^{ss}_{t+1}}{b(1-\tau^f_t)^2}+\tfrac{\gamma}{(1-\tau^f_t)}=0 \tag{23}$$

$$(\mu)\quad -\tau^{ss}_{t+1}\{\underline{\alpha}+\tfrac{(\overline{\alpha}-\underline{\alpha})}{2}(1-(1-\tau^f_t)\tfrac{b}{\gamma})\} \tag{24}$$

$$+\tfrac{\theta\beta\gamma\tau^{ss}_{t+2}}{(1-\tau^f_{t+1})b}-\tfrac{\gamma}{1-\tau^f_{t+1}}+\gamma+\gamma\ln\tfrac{1}{1-\tau^f_{t+1}}=0$$

The first condition is actually redundant in determining the equilibrium, which can be obtained from equations (23) and (24).

Equation (23) governs the choice of τ_t^f: the first term represents the cost of paying for higher subsidy levels for each birth, and the third represents the utility gains from having more children. Together these terms are the (net) deadweight loss from subsidizing fertility, and the myopic voter weights this loss against the utility gain from higher future SS benefit levels, represented by the second term.

The last condition, equation (24), is the incentive compatibility condition, and says that tomorrow's voter must be at least indifferent between keeping and abandoning SS, given the chosen tax rates.

The laws of motion for τ^{ss} and τ^f in an economy with valued SS are given by

$$\tau_{t+1}^{ss} = \frac{b}{\theta\beta}\{1 - \exp\{-\frac{\tau_t^{ss}\tilde{\alpha_{m,t}}}{\gamma}\}\} \tag{25}$$

$$\tau_t^f = 1 - \exp\{-\frac{\tau_t^{ss}\tilde{\alpha_{m,t}}}{\gamma}\} \tag{26}$$

Where $\tilde{\alpha_{m,t}}$ is an equilibrium object that maps the current SS tax to the endowment of the current decisive voter, and is obtained from the definition of $\alpha_{m,t}$ (expression (12)) and the FOC (23):

$$\tilde{\alpha_{m,t}} \equiv \underline{\alpha} + \frac{(\overline{\alpha} - \underline{\alpha})}{2}(1 - (1 - \frac{\theta\beta\tau_t^{ss}}{b})b/\gamma) \tag{27}$$

Equations (25) and (26) describe the equilibrium sequence of SS taxes and fertility subsidies respectively. Because $\frac{b}{\theta\beta}$ may be larger than one, the sequence in (25) may reach tax rates higher than one in finite time, in which case no equilibrium exists.

Here, higher future SS taxes generate incentives to increase next period tax base by increasing fertility subsidies. This version of the model then formalizes the standard argument for higher population growth in the presence of unfunded SS.

3.3.2 Rational voters

We now consider the model with voters who anticipate the effect of their choice of τ^f on the identity of the future decisive voter. The equilibrium is the solution to the same first order conditions for τ_{t+1}^{ss} and μ in the previous version (equations (22) and (24)), plus the following first order condition that governs the choice of τ_t^f:

$$-\frac{\gamma}{(1-\tau_t^f)^2}+\frac{\theta\beta\gamma\tau_{t+1}^{ss}}{b(1-\tau_t^f)^2}+\frac{\gamma}{(1-\tau_t^f)}-\mu\tau_{t+1}^{ss}\frac{b}{\gamma}\frac{(\overline{\alpha}-\underline{\alpha})}{2}=0 \qquad (28)$$

Note that the first three terms are the same as in the previous example (equation (23)). The fourth term adds a further, novel, effect from increasing τ_t^f. As fertility in one period increases, the endowment level of tomorrow's decisive voter household will also increase. Because the young at $t+1$ now form a larger constituency, a larger proportion of them will be needed to form a majority pro-SS together with the old. By Lemma 5, the decisive voter in the next period will now be a household with a higher endowment level than before. Because SS is redistributive, a higher endowment decisive-voter household at $t+1$ obtains lower net gains from participating in SS, so she will be indifferent between maintaining or abandoning the SS system at a *lower* SS tax τ_{t+1}^{ss} for each level of τ_{t+2}^{ss}.

Starting from a common initial condition, this effect implies that both fertility subsidies and SS taxes are lower in the economy with rational voters than in the economy with myopic voters. For fertility subsidies, the rational voter anticipates the negative effect on welfare of higher population growth, so the result is intuitive. For Social Security taxes, we can provide the following intuition: by construction, a decisive voter who is rational obtains higher welfare from choosing fertility, so she will need a lower future SS tax in order to become indifferent between keeping or abandoning the SS system.

Because a closed form solution for the equilibrium cannot be obtained in the case of rational voters, we compare the equilibria for the three models using numerical simulations. We choose two sets of plausible parameter values to illustrate what we believe are the interesting equilibria. [6] The laws of motion of SS taxes and fertility subsidies are plotted in figure 2. Figures 2A and B represent an equilibrium where SS is not viable without fertility subsidies. Figure 2A shows the law of motion for SS taxes against a forty-five degree line. The horizontal axis represents SS taxes at time t, and the vertical axis represents SS taxes at $t+1$. The crossed line shows the law of motion for the model without fertility subsidies. For these parameters, SS

[6] We choose the following parameter values for Example 1: $\beta = .15$, which implies an annual interest rate of 6.5% for 30 years. $b = .12$ and $\gamma = .56$, which implies a dependency ratio $\frac{\gamma}{b}$ of 4.7 in the absence of fertility subsidies, and $\underline{\alpha} = .4$, $\overline{\alpha} = 2.6$ for the distribution of endowments. Example 2 increases γ to .8

taxes reach values higher than one in finite time, so no SS system can be implemented. The dotted and continuous lines show the laws of motion for the models with myopic and rational voters respectively. Note that in both cases the SS tax converges to an interior steady state.

As in the two previous models, households that are too impatient, a distribution of income that is too equal, and low baseline fertility, all conspire against making SS sustainable. Here, voters tend to require ever increasing future SS benefits to value the SS system, ending with tax rates higher than one in finite time. In such a case, a SS system cannot be implemented.

Figure 2B shows the dynamics of fertility subsidies for the economy with myopic voters (dotted line) and rational voters (full line) against a forty-five degree line. The horizontal axis again represents SS taxes at time t, the state variable, and the vertical axis represents the rate of fertility subsidies. Note that the strategic effect on the identity of tomorrow's decisive voter implies that chosen fertility subsidies are lower for rational voters, as expected.

Figures 2C and D show the same dynamics for an example where SS is valued in the absence of fertility subsidies. In this case, the only steady state for all three economies is zero, and both SS and fertility subsidy systems shrink with time. [7]

Note that, since the path of SS taxes is lower with rational voters than in the myopic case, an equilibrium with valued SS is more likely to exist when voters fully anticipate the effect of subsidies on the identity of the future decisive voter. In this sense SS can be said to become more sustainable when the effect introduced in this section is internalized.

In our model, strategic setting of fertility subsidies provide a further tool which current young voters may use to improve their future SS benefits. That with higher subsidies the future decisive voter will be less prone to accept a high level of SS contributions is a purely political economy effect which operates besides the well understood effect of a higher tax base.[8]

---

[7]For alternative parameterizations, we found that the ranking of taxes across models remains unchanged, but the concavity of the law of motion with fertility subsidies is not guaranteed.

[8]The limit case of this effect being dominant does not imply in any way the demise of the SS system. Rather, it implies that current voters -anticipating this effect- may choose to vote for net taxes to fertility, to shape a future constituency from which they can extract higher contributions.

4 Conclusion

In this paper, we developed a political economy model of Social Security and fertility subsidies, where young generations confront promises made previously by older generations and in turn promise themselves future levels of SS benefits. A trigger strategy where SS is not reinstalled after being abandoned, sustains the voting equilibria where SS is valued.

We highlighted that, when choosing to subsidize fertility, young generations not only increase the tax base in the future but at the same time also limit their own political influence. Strategic setting of subsidies to fertility to change the identity of the future decisive voter imply that both fertility subsidies and SS taxes are lower than otherwise. In other respects, we find that fertility subsidies can sustain Social Security in cases where it would otherwise have to be abandoned, mimicking the effects of other, better understood policy tools such as public education and migration.

While the strategic effect of fertility subsidies has received little attention in the context of the sustainability of SS, there is at least one example where such subsidies have been implemented to affect the identity of the future decisive voter. Winckler [2003] describes pro-natalist policies in Israel. From 1970 to 1996, family allowances were largely conditional on family members serving in the army, effectively discriminating against the Arab-Israeli population. Such policies were clearly motivated to maintain a 'demographic balance' between Jews and Arabs, as discussed in Portuguese [1998], Friedlander [1973], and expressed in the political arena most eloquently by David Ben Gurion (see Friedlander and Goldscheider [1979]).

Our paper complements an existing literature which suggests, as in Sinn [1997], that the SS sustainability problem could be addressed by linking pension benefits to fertility. We find that such reforms are likely to be politically sustainable. Moreover, even though the political economy effect we introduce dampens the positive effect of fertility on SS, the possibility of strategically manipulating this effect makes the fertility subsidy program an even more powerful tool to help sustain an unfunded SS program.

References

Michele Boldrin and Ana Montes. The intergenerational state. Education and pensions. Technical report, University of Minnesota, December 2002.

Michele Boldrin and Aldo Rustichini. Political equilibria with Social Security. *Review of Economic Dynamics*, 3:41–78, 2000.

Georges Casamatta, Helmuth Cremer, and Pierre Pestieau. The political economy of Social Security. *Scandinavian Journal of Economics*, 102(3): 503–522, 2000.

Ignacio Conde-Ruiz and Vincenzo Galasso. Positive arithmetic of the welfare state. Technical report, FEDEA, March 2003.

Thomas F. Cooley and Jorge Soares. A positive theory of Social Security based on reputation. *Journal of Political Economy*, 107(107):135–160, 1999.

Dov Friedlander. Family planning in Israel: Irrationality and ignorance. *Journal of marriage and the family*, 35(1), 1973.

Dov Friedlander and Calvin Goldscheider. *The population of Israel*. New York, 1979.

Vicenzo Galasso. The U.S Social Security System: What does political sustainability imply? *Review of Economic Dynamics*, 2:698–730, 1999.

Vicenzo Galasso and Paola Profeta. The political economy of Social Security: a survey. *European Journal of Political Economy*, 18:1–29, 2002.

John Hassler, Jose Vicente Rodriguez Mora, and Kjetil Storesletten. The survival of the welfare state. *American Economic Review*, 93(1):87–112, 2003.

Alexander Kemnitz. Social security, public education, and growth in a representative democracy. *Journal of population economics*, 13:443–462, 2000.

Jacqueline Portuguese. *Fertility policy in Israel: The politics of religion, gender and nation*. Westport and London, 1998.

Panu Poutvaara. On the political economy of social security and public education. wp, CEBR Copenhagen, 2003.

Antonio Rangel. Forward and backward intergenerational goods: Why is Social Security good for the environment? *American Economic Review*, 93:813–834, 2003.

Kenneth A. Shepsle. Institutional arrangements and equilibrium in multidimensional voting models. *Journal of Political Science*, 23:27–59, 1979.

Hans-Werner Sinn. The value of children and immigrants in a pay-as-you-go pension system: a proposal for a partial transition to a funded system. wp 6229, NBER, 1997.

Guido Tabellini. A positive theory of social security. *Scandinavian Journal of Economics*, 102:523–545, 2000.

Onn Winckler. Fertility transition in the Middle East: The case of the Israeli Arabs. *Israel Affairs*, pages 39–67, 2003.

A Appendix

A.1 The game

This description of the game follows Conde-Ruiz and Galasso [2003], who examine a formally similar model.

History. At time t, the public history h_t is

$$h_t = \{(\tau_1^f, \tau_2^{ss}, \lambda_1), ..., (\tau_t^f, \tau_{t+1}^{ss}, \lambda_t)\} \in H_t$$

Where H_t is the set of all possible histories at time t.

Actions. For a type-i young, or a young with endowment α_i, the set of actions is

$$a_{t,i}^y = (\tau^f, \tau^{ss}, \lambda) \in [0,1] \times [0,1] \times \{0,1\}, \ all \ i$$

For an old individual, the action set is just

$$a_t^o = \lambda$$

We identify with a_t the action profile of all individuals, young and old, at time t: $a_t = (a_t^y \bigcup a_t^o)$, where

$$a_t^y = \bigcup_{i \in \Theta} a_{t,i}^y$$

, and a_t^o is as defined above.

Strategies. A strategy for a type-i young individual at time t is a mapping from h_t, the history of the game, into the action space:

$$s_{t,i}^y : h_t \to [0,1] \times [0,1] \times \{0,1\}$$

Analogously, a strategy for an old individual at time t is

$$s_t^o : h_t \to \{0,1\}$$

We denote with s_t the strategy profile played by all individuals at time t, i.e. $s_t = (s_t^y \bigcup s_t^o)$, where $s_t^y = \bigcup_{i \in \Theta} s_{t,i}^y$.

For a given action profile at time t, a_t, let $(\tau_{t,m}^f, \tau_{t+1,m}^{ss}, \lambda_{t,m})$ be the median of the distribution of each policy choice. We call $(\tau_{t,m}^f, \tau_{t+1,m}^{ss}, \lambda_{t,m})$ the outcome function of the voting game at t.

This outcome function corresponds to the Structure Induced Equilibrium outcome of a voting game at time t in which agents can commit over the future policies.

The history of the game is updated according to the outcome function. At time $t+1$:

$$h_{t+1} = \{(\tau_1^f, \tau_2^{ss}, \lambda_1), ..., (\tau_t^f, \tau_{t+1}^{ss}, \lambda_t), (\tau_{t,m}^f, \tau_{t+1,m}^{ss}, \lambda_{t,m})\} \in H_{t+1}$$

Payoffs. For a given sequence of action profiles $(a_0, ..., a_t, a_{t+1}, ...)$ and their corresponding realizations $((\tau_1^f, \tau_2^{ss}, \lambda_1), ..., (\tau_t^f, \tau_{t+1}^{ss}, \lambda_t), (\tau_{t+1}^f, \tau_{t+2}^{ss}, \lambda_{t+1}), ...)$, the payoff function for a type-i young individual is $V_t^y(\tau_t^f, \tau_{t+1}^{ss}, \lambda_t, \tau_{t+2}^{ss}, \lambda_{t+1})$ according to equation (9). For an old agent, the payoff function is $V_t^o(\lambda_t, \tau_{t-1}^f, \tau_t^{ss})$ according to equation (10).

Let $s_{t/i}^y = s_t^y / s_{t,i}^y$ be the strategy profile at time t for the young individuals except for the type-i young individual. At time t, the type-i young individual chooses $s_{t/i}^y$ to maximize the function

$$\begin{aligned} & v_t^y(s_0, ..., (s_{t/i}^y, s_{t,i}^y), s_t^y, s_{t+1}, ...) \\ = \ & V_t^y(\tau_{t,m}^f, \tau_{t+1,m}^{ss}, \lambda_{t,m}, \tau_{t+2,m}^{ss}, \lambda_{t+1,m}) \end{aligned}$$

An old individual with identity j, in turn, maximizes the function

$$\begin{aligned} & v_t^o(s_0, ..., (s_{t/i}^y, s_{t,i}^y), s_t^y, s_{t+1}, ...) \\ = \ & V_t^o(\lambda_{t,m}, \tau_{t-1,m}^f, \tau_{t,m}^{ss}) \end{aligned}$$

Where the functions $v^{y,o}$ map strategy profiles to payoffs, and variables with subscript m are the medians among the actions over the relevant parameters.

The equilibrium. We define now a Stationary Subgame Perfect Structure Induced equilibrium of the voting game.

Definition (SSPSIE): A stationary voting strategy profile $s = \{(s_t^y \bigcup s_t^o)_{t=0}^{\infty}\}$ is a SSPSIE if the following conditions are satisfied:

1. s is a subgame perfect equilibrium.

2. At every period t, the equilibrium outcome associated to s is a Structure Induced equilibrium of the static game with commitment over future policy.

A.2 The choice of taxes is independent of individual endowments

Once the SS system has been continued, the tax rates are chosen by solving:

$$\max_{\{\tau_t^f,\tau_{t+1}^{ss}\}} \quad V^y(\tau_t^f, \tau_{t+1}^{ss}, \tau_t^{ss})$$

$$= (1-\tau_t^{ss})\alpha_i + \tfrac{\theta\beta\gamma\tau_{t+1}^{ss}}{(1-\tau_t^f)b} - \tfrac{\gamma}{1-\tau_t^f} \quad +\gamma \ln \frac{\gamma}{(1-\tau_t^f)b} \tag{29}$$

$$s.t. \quad -\tau_{t+1}^{ss}\alpha_{t+1}^m + \tfrac{\theta\beta\gamma\tau_{t+2}^{ss}}{(1-\tau_{t+1}^f)b}$$

$$-\tfrac{\gamma}{1-\tau_{t+1}^f} + \gamma + \gamma \ln \tfrac{1}{1-\tau_{t+1}^f} = 0 \quad \text{if } \tau_t^f > 1-\gamma/b \tag{30}$$

$$\tau_{t+1}^{ss} \leq 1 \quad \text{otherwise} \tag{31}$$

Where α_{t+1}^m is a function of n_t (equation (12). This problem is equivalent to the modified problem where $(1-\tau_t^{ss})w\alpha_i$ is eliminated from the objective function:

$$\max_{\{\tau_t^f,\tau_{t+1}^{ss}\}} \quad V^y(\tau_t^f, \tau_{t+1}^{ss}, \tau_t^{ss}) - (1-\tau_t^{ss})w\alpha_i$$

$$= \tfrac{\theta\beta\gamma\tau_{t+1}^{ss}}{(1-\tau_t^f)b} - \tfrac{\gamma}{1-\tau_t^f} \quad +\gamma \ln \frac{\gamma}{(1-\tau_t^f)b} \tag{32}$$

$$s.t. \quad -\tau_{t+1}^{ss}\alpha_{t+1}^m + \tfrac{\theta\beta\gamma\tau_{t+2}^{ss}}{(1-\tau_{t+1}^f)b}$$

$$-\tfrac{\gamma}{1-\tau_{t+1}^f} + \gamma + \gamma \ln \tfrac{1}{1-\tau_{t+1}^f} = 0 \quad \text{if } \tau_t^f > 1-\gamma/b \tag{33}$$

$$\tau_{t+1}^{ss} \leq 1 \quad \text{otherwise} \tag{34}$$

Because the endowment (α_i) does not play a role in this problem, the solution cannot be a function of it.

A.3 Derivation of the Equilibrium for the model with myopic voters

In an interior solution, the equilibrium is obtained by maximizing the objective function in (20) subject to the constraint in (21):

$$\max_{\{\tau_t^f,\tau_{t+1}^{ss}\}} \quad V^y(\tau_t^f, \tau_{t+1}^{ss}, \tau_t^{ss})$$

$$= (1-\tau_t^{ss})\alpha_i + \tfrac{\theta\beta\gamma\tau_{t+1}^{ss}}{(1-\tau_t^f)b} - \tfrac{\gamma}{1-\tau_t^f} + \gamma \ln \tfrac{\gamma}{(1-\tau_t^f)b} \tag{35}$$

$$\begin{aligned} s.t. \quad -\tau_{t+1}^{ss}\{&\overbrace{\frac{(1-(1-\tau_t^f)b/\gamma)(\overline{\alpha}-\underline{\alpha})}{2}+\underline{\alpha}}^{\alpha_{m,t+1}}\} + \tfrac{\theta\beta\gamma\tau_{t+2}^{ss}}{(1-\tau_{t+1}^f)b} \\ &-\tfrac{\gamma}{1-\tau_{t+1}^f} + \gamma + \gamma\ln\tfrac{1}{1-\tau_{t+1}^f} = 0 \end{aligned} \tag{36}$$

To derive constraint (36), note that it represents the condition

$$V_m^y(\tau_{t+1}^f, \tau_{t+2}^{ss}, \tau_{t+1}^{ss}) \geq V_m^y(0,0,0) \tag{37}$$

Where the indirect utilities are those of the decisive voter at time $t+1$ and the arguments in the function on the left hand side are chosen optimally. From (9), this expression is equivalent to

$$(1-\tau_{t+1}^{ss})\alpha_{t+1}^m + \tfrac{\theta\beta\gamma\tau_{t+2}^{ss}}{(1-\tau_{t+1}^f)b} - \tfrac{\gamma}{1-\tau_{t+1}^f} + \gamma\ln\tfrac{\gamma}{(1-\tau_{t+1}^f)b} \tag{38}$$

$$\geq \quad \alpha_{t+1}^m - \gamma + \gamma\ln\tfrac{\gamma}{b} \tag{39}$$

Which, after some algebra, leads to condition (36).

To obtain the first order conditions (22) to (24), we take the expression for $\alpha_{m,t+1}$ in the constraint (36) as parametric, which affects the derivation of the FOC for τ_t^f, expression (23).

From (23) we obtain:

$$\tau_t^f = \tau_{t+1}^{ss}\frac{\theta\beta}{b} \tag{40}$$

Note that equation (24) can be written as

$$\begin{aligned} &-\tau_{t+1}^{ss}\{\underline{\alpha} + \tfrac{(\overline{\alpha}-\underline{\alpha})}{2}(1-(1-\tau_t^f)\tfrac{b}{\gamma})\} \\ &+\tfrac{\gamma}{1-\tau_{t+1}^f}(\tfrac{\theta\beta}{b}\tau_{t+2}^{ss} - \tau_{t+1}^f) + \gamma\ln\tfrac{1}{1-\tau_{t+1}^f} = 0 \end{aligned} \tag{41}$$

The second term of this expression is zero, which comes from leading expression (40) one period. Using the identity in (27), equation (41) becomes

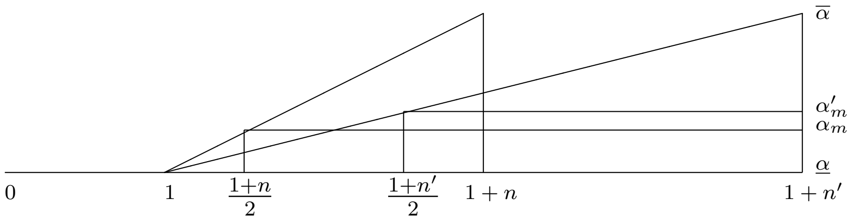
$$\tau_{t+1}^{ss}\tilde{\alpha_{m,t+1}} = \gamma\ln(\frac{1}{1-\frac{\theta\beta}{b}\tau_{t+2}^{ss}}) \tag{42}$$

This leads to

$$\tau_{t+2}^{ss} = \frac{b}{\theta\beta}\{1-\exp\{-\frac{\tau_{t+1}^{ss}\tilde{\alpha_{m,t+1}}}{\gamma}\}\} \tag{43}$$

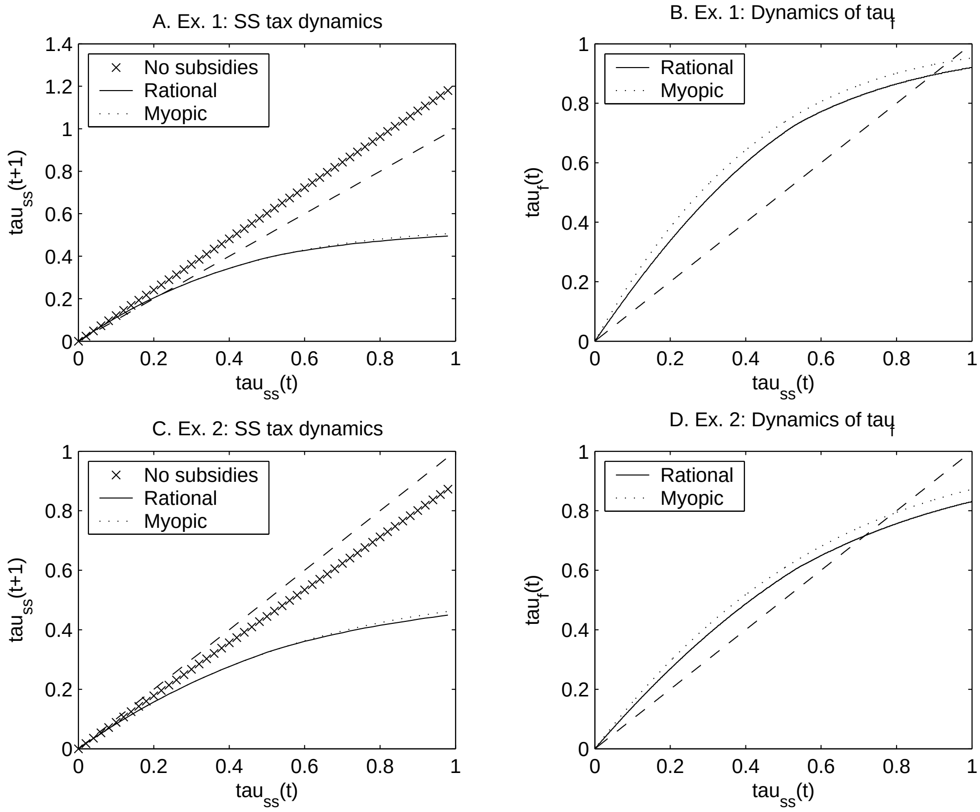
The law of motion in (25) is this same expression lagged one period. The function characterizing the evolution of fertility subsidies can be obtained from (43) and (40).

Figure 1: Who is the decisive voter



Households

Figure 2: Tax rate dynamics



$$Parameters\ Example\ 1: \quad \{\beta = .15, b = .12, \gamma = .56, \overline{\alpha} = 2.6, \underline{\alpha} = .4\}$$
$$Parameters\ Example\ 2: \quad \{\beta = .15, b = .12, \gamma = .8, \overline{\alpha} = 2.6, \underline{\alpha} = .4\}$$